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Uncertainty and the Intertemporal Forest Management of Natural Resources: An Empirical Application to the Stanislaus National Forest

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The empirical problem of natural resource management is typically the intertemporal allocation of product flows and resource stocks under uncertainty. National forest harvest scheduling is conceptualized in this study as a stochastic optimal control problem. In theory, optimal solutions to most stochastic control problems exist, but required computer costs are excessive even for problems of moderate dimensions. Hence, approximate solution techniques are required, and this study employs one such approach called the Linear–Quadratic–Gaussian (LQG) control method.

Given a set of desired or target levels for stocks and flow variables, the LQG optimization criterion is to keep the actual evolution of the system close to the target levels. Optimal harvest levels are given as the solution to a simplified model of the actual problem where the model is characterized by quadratic preferences and linear system dynamics. Imperfect observation of timber stocks is also modeled in the LQG approach, and part of the solution is a recursive estimator of timber levels that is the optimal estimator given management objectives. The relative costs of uncertainty are also calculated in the LQG solution. Empirical results isolate those sources of uncertainty critical to management actions and indicate superiority of recursive estimation over static estimation.

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1. INTRODUCTION

Managers of replenishable natural resources are faced with a stochastic intertemporal optimization problem. The sources of uncertainty for the specific case of national forests can be divided into three categories. The first involves uncertainty about the actual forest dynamics, including the evolution of forest resource stocks over time and the impact of management actions and random disturbances upon that evolution. The second category is uncertainty in estimating the extent of resource stocks. Current U. S. Forest Service policy is to take a decennial survey to estimate timber stocks. The third category of uncertainty is specification of the preference function and its parameters. This category encompasses not only the underlying objectives (such as maximizing profit or social benefit) but also the parameters that measure those goals (such as prices, costs, or social benefits per unit of good produced).

Since uncertainty enlarges the management problem, actions that reduce uncertainty should be considered and perhaps taken by the manager. Such consideration cannot be pursued independently of production decisions. Uncertainty usually reduces returns from an enterprise. The extent of reduction will depend on the form of the decision–maker's preferences and on how directly a source of uncertainty affects a given production process. In all but the most special of problems, optimal production levels change when uncertainty is explicitly recognized in formulating a problem.

The current national forest management procedure is to allocate national forestlands according to various uses, such as wilderness preserve or timber production. Plans are then generated to determine management actions on the classified lands. This procedure largely ignores uncertainty, assuming implicitly that using the means of random variables will yield essentially optimal plans. This is the current procedure because, for such a large–scale problem, explicit recognition of uncertainty would make the problem impossible to solve with today's computers. Since an optimal solution to the exact stochastic problem is too costly or impossible to obtain, an approximate solution technique must be used.

Numerical methods are required to solve the general form of stochastic intertemporal optimization problems. Even problems of very modest dimensions entail computational costs that rapidly become excessive. Hence, many natural resource production problems are solved by assuming away uncertainty. Engineers, encountering stochastic problems mathematically similar to those in resource production, have developed approximate methods that explicitly recognize stochastic behavior and have computationally efficient solutions. One such method, the Linear–Quadratic–Gaussian (LQG) control model, has been used in aerospace guidance systems and has been adapted to some economic problems.

A two–stage optimization process is used in this study. The first stage consists of solving a deterministic version of the optimization problem, yielding a time series of target levels for the system variables to attain. The second stage involves using the LQG solution to adjust managerial (control) actions as the system is actually evolving through time, allowing the system variables to be kept close to their predetermined target levels. Thus, without directly optimizing the underlying stochastic managerial problem, the technique offers an approximate method for stochastic optimization.

Potentially, use of the LQG model by a manager or policymaker can yield several types of information relevant to decision making. First, the optimal actions to take at a given time can utilize all available information since the decisions are made sequentially. Second, derivation of the expected value of the objective function yields the costs of several sources of uncertainty. The costs are related directly to the policy being implemented. Third, the LQG employs an estimator of the current state of the system that is optimal with respect to the management goals. To obtain the above types of information, the underlying optimization problem must be amenable to representation as an LQG problem.

This study was undertaken to determine how successfully the LQG approach could be used as an aid to planning timber management. Required first is the examination of both natural resource production models and the basis of the LQG optimization philosophy. An empirical case is then presented to determine how readily a timber management problem can be solved with the LQG, whereupon the empirical usefulness of the LQG for timber management can be appraised. On a broader scale, this study also investigates aspects of the LQG method that have inherent difficulties for application to economic and, particularly, natural resource models.

The results of the study show that the LQG methodology can be usefully employed to solve empirical, stochastic, and natural resource management problems, thus yielding the user costs of deviations from targeted resource stocks and product flows. The values of improved precision in the estimates of the resource levels and production dynamics are also computed in terms of the cost of target deviations. The use of the Kalman filter approach to estimate the available resource stocks is shown to have considerable advantages over the conventional periodic sampling approach. For the model estimated and then simulated in this study, the recursive estimates reduce the variances of the stock estimates by about one–half. The analysis of all of the sources of imprecision in this specific empirical study shows that improvement of the estimated relationship between timber volume and basal area would enhance both the optimal control actions and the precision of the estimates of timber stocks.

The study is dichotomized into two major parts—methodological and empirical. Each part has several sections. Section 2, following this first introductory section, reviews the principles of natural resource production in a deterministic environment. It also discusses the nature of the changes thrust upon the decision–maker when uncertainty is introduced. Also considered are appropriate management strategies in a stochastic environment such as maximizing expected utility, acquisition of better information, and perturbations imposed on the system in order to obtain better information. The burden of computational constraints is examined for justification of the use of an approximate strategy. Section 3 develops the LQG optimizing technique from both philosophical and pragmatic points of view. It includes a detailed exposition on the use of the LQG and a brief survey of problems that interfere with the use of the LQG for natural resource production problems.

The next two sections delve into an empirical application of the methodology developed in the two prior sections. Section 4 presents the estimation and construction of an LQG model of harvest activities on a mixed–conifer timber type that is in the standard component (the main area allocated to timber production) of the Stanislaus National Forest. The species mix in this study is limited to permit inexpensive experimentation with the model, and harvest activities are modeled because timber production is a major concern in national forest management. Since the LQG technique models management objectives in addition to resource growth dynamics, an analysis is made of current management practices and policies on the national forests. Additionally, the acceptability of the LQG growth dynamics is appraised. Section 5 analyzes the results of several simulations under various stochastic assumptions and policies. This analysis reveals the extent to which the LQG technique can be used as an aid to forest management under uncertainty. Section 5 emphasizes optimum harvest levels under uncertainty, estimation of inaccessible timber stocks, and evaluation of the costs of uncertainty.

Section 6, the final section, summarizes both the implications of the particular empirical application and more general methodological questions concerning the use of the LQG approach for natural resource management that are raised in the empirical application.

2. NATURAL RESOURCE PRODUCTION UNDER UNCERTAINTY

The method for scheduling production under uncertainty that is considered in this study is suitable for many objectives and types of resource production. Since the method utilizes production levels given by a solution to a deterministic model in the optimization process and the approximate stochastic model minimizes deviations about these levels, Sections 2.2 and 2.3 briefly summarize a deterministic model and necessary conditions for optimal production rates, respectively. Section 2.4 then extends the deterministic model to include uncertainty and considers optimal strategies in response to various sources of uncertainty. Section 2.5 discusses the computational aspects of optimal stochastic solutions as criteria for approximation. Given the wide array of topics included in natural resource economics, the areas of resource theory considered in this study must be defined. This is done in Section 2.1.

2.1. Definitions and Assumptions Underlying the Study

Resource production is defined in this study as (1) the transformation of a given natural resource stock into a flow of goods or services to be consumed directly or used as inputs to subsequent goods and services and (2) activities to upgrade the resource stock. The present study is concerned with natural resource production at the firm level within a given geographical area, and the general management policies are given by legal mandate. Since such guidelines can rarely be put into a mathematical form suitable for use in an optimization model, it is assumed that the decision–maker completes the preference structure to yield a usable objective function for model optimization. The empirical aspects of such a construction for national forests are considered in 4.2.

Given that the legal mandates suggest policies that would differ from those derived from profit maximization, the consideration of nonpecuniary externalities becomes relevant. Such consideration may require measuring such things as option demand and the effects of irreversible resource changes. In most stumpage–oriented, harvest–scheduling models, a zero value is assigned to the present use of standing stocks not harvested in the current period, implying that no current utility results simply from their existence. By modeling natural resource problems as optimal control problems, it is possible to include some external effects associated with the level of the resource stock. This is accomplished by recognizing explicitly the current utility of resource stocks in the management criterion function.

In summary, this study is concerned with a microeconomic analysis of a publicly owned natural resource on the firm level. Production flows are determined from legally mandated policies given in an amorphous form that represent society's preferences. Given the above limitations, resource production in a deterministic environment is now considered.

2.2. A Firm Model of Resource Production

Resource production is typically viewed as the balancing of current utility from a resource stock against future utility, given biological constraints on stock evolution. Utility or profit is generated by the output from the resource, u_t^o, that requires stock transformation plus the simple existence of the stock, although distinction between the two is sometimes arbitrary. For lack of a better name, the latter value is labeled the stock's temporal existence value. As an example, a standing forest in a recreational society generates utility by existing and being accessible. A resource production function can be written, letting x_t be a vector of resource stocks (states) and u_t a vector of inputs and outputs (controls) as:

$$x_{t+1} = F_t\ (x_t, u_t) \qquad t = 0, \ldots, T-1 \tag{1}$$

where

$$u_t' = \left(u_t^o, u_t^i\right)$$

u_t^o = vector of resource outputs

and

u_t^i = vector of inputs to improve or alter the resource stock.

Relationship (1) is frequently referred to as the system dynamics.[1]

Given the hypothesized relation of u_t to the variable input, it is then possible to construct a profit or social welfare function of the form

$$J = \sum_{t=0}^{T-1} \beta_t\ (x_t, u_t) + \beta_T\ (x_T) \tag{2}$$

where J is assumed to be concave. To illustrate the role of temporal existence values, assume for the moment that J is a profit function of the form

$$\pi_t = P_{t+1}\ (x_{t+1})'\ x_{t+1} + P_t\ \left(u_t^o\right)'\ u_t^o - M_t\ \left(u_t^o\right)'\ u_t^o - M_t\ \left(u_t^i\right)'\ u_t^i$$

where P(•) and M(•) are vectors of price and cost, respectively. They are constructed so that price can be a function of the quantity marketed, and the cost of u_t can be increasing or decreasing.

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[1]As specified, (1) does not have any capital stocks. These can be added by partitioning x_t to include both resource and capital stocks. The functional form, F_t, would become more complex. Since capital stocks do not enter directly into the empirical application, they are not explicitly considered here.

The function P(x) is the temporal existence value of a unit of the stock of resources. It does not include the imputed value of the resource stock as a source of future streams of output, $U^{O}_{t+i}, \forall i \geqslant 1$. This latter value is the future profitability that can be attributed to x_t as a source of future output. Scott (1967) defines this value as the user cost of consuming resources today, *i.e.*, foregone future profitability or utility. When $P(x) \neq 0$, it also becomes a component of user cost as is shown analytically in 2.3.

Given the form of (1) and (2), the basic problem of resource economics is equivalent to an optimal control problem which can be solved by various methods discussed in 2.3. In addition, there are certain restrictions on (1) and (2) that (2) is assumed to be separable with respect to time, that (1) and (2) are each differentiable at least once, and that (2) satisfies the Kuhn–Tucker constraint qualification. Since most timber management models are in discrete time, the discrete approximation is used accordingly throughout this study.

The control or natural resource production problem can be written compactly as:[1]

$$\text{Max } J = \sum_{t=0}^{T-1} \beta_t \ (x_t, \ u_t) + \beta_T \ (x_T) \tag{3}$$

subject to

$$x_{t+1} = F_t \ (x_t, \ u_t) \tag{4}$$

$$u_t \in \{U\}, \qquad x_0 = x(0). \tag{5}$$

The set denoted by U is assumed to be convex and contains all feasible values for the controls, and x(0) is the known initial state or stocks of resources.

2.3. Optimal Rates of Production Under Certainty

The necessary conditions for optimality in the above problem can be developed by dynamic programming or the use of the maximum principle of Pontryagin *et al.* (1962).[2] The latter approach offers a more enlightening interpretation although––like the Kuhn–Tucker conditions––it provides no explicit algorithm for obtaining a solution, whereas dynamic programming is a solution technique.

The actual Pontryagin conditions were developed for the control problem in continuous time, and they have subsequently been derived for discrete problems by Athans (1968), among others. One such approach is to develop (3)–(5) as a problem using Lagrange multipliers; then the Pontryagin conditions become identical to the Kuhn–Tucker conditions. Following Burt and Cummings (1970), (3)–(5) can be rewritten as

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[1]The objective functional (3) is composed of two terms: the intermediate function and the terminal–value or final–state function, β_T (x_T). A "functional" is defined in Baumol (1970) as a function whose domain comprises a set of functions.

[2]Additionally, the optimal solution could be obtained using classical mathematical programming as discussed in Canon, Cullum, and Polak (1970).

$$L = \sum_{t=0}^{T-1} \beta_t\ (x_t,\ u_t) + \beta_T\ (x_T) - \sum_{t=0}^{T-1} \lambda'_{t+1}\ [x_{t+1} - F_t\ (x_t,\ u_t)]$$
$$- \sum_{t=0}^{T} \sum_{i=1}^{c} \gamma_{t,i}\ [g_{t,i}\ (x_t,\ u_t) - k_{t,i}] \tag{6}$$

$$g_{t,i}\ (x_t,\ u_t) = k_{t,i} \qquad i = 1, \ldots, j - 1$$
$$g_{t,i}\ (x_t,\ u_t) \geqslant k_{t,i} \qquad i = j,\ j + 1, \ldots, c. \tag{7}$$

The last term in (6) includes both strict equality and inequality constraints on the state and control variables. For $g_{t,i}$, $i < j$, the constraint is a strict equality. For $g_{t,i}$, $i \geqslant j$, it is assumed that inequality constraints apply. The $k_{t,i}$ define the boundary of the feasibility set. Note that it is through the $g_{t,i}$ that any nonnegativity constraints are placed on the x_t, u_t.

A complete derivation and interpretation of the necessary and sufficient conditions are given in Burt and Cummings (1970). To lay a groundwork for understanding stochastic control problems and appraising approximate solutions, observe that the optimal rate of production is

$$\frac{\partial L}{\partial u_t} = \frac{\partial \beta_t}{\partial u_t} + \frac{\partial F'_t}{\partial u_t} \lambda_{t+1} - \frac{\partial g'_t}{\partial u_t} \gamma_t = 0 \qquad t = 0, \ldots, T - 1 \tag{8}$$

where λ_{t+1} gives the marginal value of an additional unit of x_{t+1} over the remainder of the time horizon and is derived recursively from

$$\frac{\partial L}{\partial x_t} = \frac{\partial \beta_t}{\partial x_t} - \lambda_t + \frac{\partial F'_t}{\partial x_t} \lambda_{t+1} + \frac{\partial g'_t}{\partial x_t} \gamma_t = 0 \qquad t = 0, \ldots, T - 1. \tag{9}$$

Thus, λ_t is the user cost of a natural resource which Scott defines as ". . . the present value of future profit foregone by a decision to produce a unit of output today" (Scott, 1967, p. 42). Recall that it is argued in 2.2 that user cost is a function of future consumption streams as well as its temporal existence value which is represented by $\partial\beta_t/\partial x_t$ in (9). By solving (9) for λ_t, it can be seen that user cost is partially a function of the existence value of a resource. In a competitive industry with complete futures markets, λ_t would be the price of a unit of resource.

Given the interpretation of λ_t, (8) can be rewritten as follows:

$$\frac{\partial \beta_t}{\partial u_t} - \frac{\partial g'_t}{\partial u_t} \gamma_t = -\left(\frac{\partial F'}{\partial u_t}\right) \lambda_{t+1}. \tag{8a}$$

Assuming $\partial F_t/\partial u_t^o < 0$, the rule for resource production is to increase output in the present until the marginal social benefit less boundary costs is just equal to the user cost.

Necessary conditions (8) and (9) only verify whether a given solution is optimal; they do not provide a solution algorithm. The system (3)–(5) can also be solved by dynamic programming. To illustrate the use of dynamic programming, the recurrence relation for the solution is written:

$$\Lambda_t\ (x_t) = \max_{u_t}\ [\beta_t\ (x_t,\ u_t) + \Lambda_{t+1}\ (x_{t+1})] \tag{10}$$

where the function $\Lambda_{t+1}\ (x_{t+1})$ gives the optimal action for whatever value of x_{t+1} results as a consequence of the action, u_t, taken in period t. The term Λ_t is the total value of the objective functional for the problem beginning with x_t. From (10), it is seen to be composed of two parts: the intermediate function giving current profit, $\beta_t\ (x_t,\ u_t)$, and the profit that will accrue to x_{t+1} over the remainder of the planning horizon, Λ_{t+1}.

Equation (10) solves for the optimal controls using backward recursion; that is, the solution is to calculate the optimal u_{T-i} given x_{T-i} ($i = 1, \ldots, T$). Burt and Cummings (1970) point out that this is a Markovian dependence structure in the decision process. From (10), it can be observed that, once x_t is known, the u_{t-i}, $i = 1, 2 \ldots$ have no effect on u_t. Hence, the current state variables completely summarize the effects of all prior controls.

In the problem where parameters are known with certainty, the entire trajectory of control variables can be derived at $t = 0$. This is called a pure open–loop strategy. In stochastic problems such decision strategies are inefficient. A sequential decision process yields a higher expected return, as discussed by Charnes, Dreze, and Miller (1966), since current decisions are computed with the use of all available information. The difficulties introduced by the inclusion of uncertainty are discussed further in 2.4.

In summary, the necessary conditions for optimal production of resources require that the marginal benefits of current consumption be just equal to the marginal profitability of having that unit of resource in the future. These conditions are obtained via the discrete maximum principle, though they also can be derived using dynamic programming. Dynamic programming is used also for stochastic problems which suggests, at this point in the study, that some strong parallels exist between the necessary conditions for deterministic and stochastic problems.

2.4. Natural Resource Production in a Stochastic Environment

In this section the assumption of perfect knowledge of the system and goals is relaxed. Several sources of uncertainty that arise in natural resource models are classified and the resulting optimization problems considered.

For natural resource control models, stochastic behavior arises in three ways:

1. Uncertainty about the structure and parameters of the objective function often exists.

2. There is uncertainty about the mathematical form of the system dynamics. Given the exact form of the dynamics, there may be uncertainty about the exact parameter values.

3. The state variables or resource stocks of the system usually cannot be observed perfectly. Since each type of uncertainty presents distinct problems, they are discussed in three separate sections.

2.4.1. Uncertain Rewards and Optimal Strategies With Respect to Uncertainty

In many empirical natural resource problems, the objective function parameters are uncertain to the extent that the exact demand curves may not be known or there is uncertainty about the rates of substitution between resource uses in society's preferences. Additionally, the decision–maker must determine some strategy to cope with uncertainty. Thus, two problems are discussed in this section: (1) the uncertainty about the parameters of the objective function and (2) the general problem of selecting a strategy for optimization regardless of the source of uncertainty.

In this study it is assumed that the decision–maker combines the legal mandates and his perceptions of the wishes of society into a social utility function and then attempts to maximize this function. In a sense this means that the decision–maker acts as an agent to combine and order the conflicting wants of society into a scalar–valued function. Realistically, he is not sure that the utility function, so constructed, is the one that will maximize utility; but, at some point, some agent must decide which set of outcomes is preferred to the others. Uncertainty about the utility function can be represented by modeling parameters as random variables. Alternatively, a set of preference functions can be constructed and optimal solutions computed for each structure to identify the structure or combination of structures that is optimal. This latter approach is discussed further in 4.3.1 and in Rausser and Freebairn (1974).

The presence of some form of uncertainty in the problem requires the adoption of a strategy or posture toward uncertainty. In this study the control problem is optimized by maximizing expected utility. As discussed in Dillon (1971), maximizing utility by maximizing expected utility is what a decision–maker should do whose preferences satisfy ordering, continuity, and independence. Maximizing expected utility provides the foundation for the similarity between the necessary and sufficient conditions in deterministic and stochastic problems. In optimizing a stochastic problem, the expectations operator is first used on the random components of the problem to express the mean value of these components in terms of their moments. Thus, the optimal stochastic necessary conditions can be developed in a form that is analogous to conditions (8) and (9). The optimal solution to a sequential stochastic problem will differ from deterministic solutions because the optimal actions are determined on a sequential basis. Dynamic programming is usually employed in stochastic problems since evaluation of the expected value is usually straightforward in a dynamic programming formulation.

The decision–maker's range of possible actions increases in problems characterized by uncertainty because additional information or data can be obtained to lessen uncertainty. Such acquisitions can usually be classified either as passive or active learning. The latter form of information acquisition is applicable only to sequential decision–making processes and is discussed more fully in Section 2.4.2. Passive learning generally involves the purchase of additional information. Such information should be acquired when the reduction in uncertainty less the cost of the information results in a net gain in expected utility.

To summarize, a public resource manager is in the position of having to make a continual series of decisions under uncertain circumstances. The decision set includes both decisions on the levels at which to set controls and actions to be taken to reduce uncertainty. These two types of action are not always independent of one another as discussed below; but, in either case, the decision–maker maximizes social utility by maximizing expected social utility given the assumptions on the decision–maker's behavior.

2.4.2. Uncertainty in the System Dynamics

In this section the problems posed by uncertainty about the parameters of the system dynamics are considered. Letting w be a vector of both underlying random behavior and unknown population constants, the system dynamics (4) can be rewritten:

$$x_{t+1} = F_t^* \ (x, \ u, \ w). \tag{4a}$$

The distinction between random parameters and population constants is important since it influences the choice of estimation technique. Classical sampling theory posits that the components of w are population constants as in ordinary least squares. A different approach is treating the w_i as random variables (Swamy and Mehta, 1973).

Under both the classical and error component specifications, estimation precision is typically increased by additional sample information. Often the manager of a particular resource cannot purchase passive information about those aspects of production that are unique to the particular physical assets and economic circumstances. For sequential decisions, however, new observations of data become available in each time period so that the parameter estimates can be updated at each decision point. The optimal decision for the current problem can be determined with the use of the updated parameters. When the parameter estimates are updated at each decision point, the control problem has a dual nature. The levels of the optimal controls affect both the current value of the objective function and the quality of future information about w. For example, let the system dynamics (4) be represented by the following scalar equation:

$$x_{t+1} = ax_t + bu_t + e_t \tag{11}$$

where

$$E(e_t) = 0$$

$$E\ (e_t\ e_t) = \sigma$$

and a and b are unknown. Letting the objective function (3) be known and the quadratic in x_t and u_t and using Bayesian methods, MacRae (1972) shows that the conditional covariance of (a_t, b_t), Γ_t, is given by

$$\Gamma_t^{-1} = \Gamma_{t-1}^{-1} + \frac{(x_{t-1}, u_{t-1})' \ (x_{t-1}, u_{t-1})}{\sigma} \tag{12}$$

and the conditional mean by

$$(\bar{a}_{t+1}, \bar{b}_{t+1}) = \Gamma_{t+1} \left[\Gamma_t^{-1} \ (\bar{a}_t, \bar{b}_t)' + \frac{(x_t, u_t)' \ x_{t+1}}{\sigma} \right]. \tag{13}$$

Aoki (1967) shows that this problem cannot be solved analytically but requires instead numerical methods.

The above problem is called the active learning control problem. It is analytically intractable because control levels interact with future values of the parameters as illustrated in (12) and (13). The decision–maker has a dual problem in these circumstances. From (12) and (13), the level of controls affects both the degree of precision on future parameter estimates and the current returns measured by the objective function. Since higher levels of precision may lead to higher expected values, it is frequently in the interest of the decision–maker to sacrifice some current control effort (current benefits or profits) to learn more about the system parameters. The adaptive control problem is a design–of–experiments problem and a control–determination problem, both inextricably related. It should be emphasized that, in a dynamic problem where the exact density of the parameters is unknown, maximizing the expected utility of the problem *requires* at least consideration of using an active learning strategy. Any other solution is suboptimal.

2.4.3. Estimating the Level of Resource Stocks

The problem for the resource manager of determining the level of existing resource stocks is both physical and economic. The resource may be quite literally inaccessible, eliminating the possibility of perfect measurement as with mineral or petroleum deposits. The cost of determining the exact quantity or quality of the resource may be prohibitive in the sense that exact assessment of the extent of resource stocks would cost more than the resource enterprise would ever be worth. This is particularly true if a continually current estimate of resource stocks is desired. Such an estimation problem is referred to in engineering literature as a filtering problem. Specifically, Gelb (1974) defines filtering as estimating the current level of the state vector given all past measurements.

Filtering methods require the use of an observer relationship which can be added to the control problem specification (3)–(5) and represented as:

$$z_t = \ell_t (x_t) + v_t \tag{14}$$

where z_t is the observed data and v_t is a random error vector usually assumed to have expectation zero and a finite variance that indicates observer or sampling precision.

In a control problem the only data available are assumed to be $x_0, z_1, \ldots, z_t$ and $u_0, u_1, \ldots, u_{t-1}$. Thus, the filter combines this information in some fashion to determine an estimate of x_t that is optimal with respect to the control problem. The exact form of the filter depends on the algebraic and stochastic characteristics of the system dynamics (4) and the observer relationship (14).

Observations are costly, and the manager must decide how precise the observation process should be. Specifically, if the variance of v_t is zero, (14) represents a perfect observer. In many instances the manager can control the variance of v_t by either using more precise technologies or increasing the sample size. A detailed model for selecting optimal sample size is given in Rausser and Howitt (1975). Only under fairly stringent assumptions can a stochastic control problem be solved to yield a solution to both the optimal control problem and the setting of optimal sampling intensity.

2.5. Computing Optimal Solutions to Stochastic Control Problems: Difficulties and Approximations

The above discussion places very few restrictions on the functional form of the objective function or the system dynamics. A solution to the deterministic problem can be derived by dynamic programming. In practice, this proves quite difficult and frequently impossible. For all but the most specific of algebraic forms, computer storage requirements for solving even a moderately sized problem can be excessive.

When the deterministic assumption is relaxed, obtaining an optimal solution becomes even more difficult. Assuming that the uncertainty is of a type requiring an active learning strategy for an optimal solution, a numerical approximation almost inevitably has to be made. Further complications arise when the noisy observation problem is included as part of the optimization problem. Once again, most empirical problems require an approximate filter (estimator).

Given that the optimal solution to the exact problem cannot be obtained, approximate procedures must be employed. Many methods exist for making approximations such as assuming away uncertainty or selecting arbitrary decision rules. The best feasible approximate technique for stochastic control problems is difficult to select because no truly optimal solution can be computed with which approximate solutions can be compared.

An obvious criterion for selecting a particular approximate solution technique is that the approximate problem closely resembles the actual problem. Other criteria become important when the purposes of developing a model are considered. From the viewpoint of social or economic planning, it is important that the technique generate answers at a low cost since extensive experimentation is usually required to both validate the model and determine which policies give the "best" performance. Computational ease is also important since it is often of interest to determine the implications of various assumptions or changes in the model structure.

This study is particularly concerned with the effect of uncertainty on natural resource management. Questions that need to be answered are: What sources of uncertainty are the most costly? How do these costs vary with changes in the management objective function? What is the impact on production levels of assuming that parameters are known when they are really unknown? What is the best method of combining and utilizing sample information? Does neglecting the active information aspect of stochastic optimization give vastly suboptimal results?

The above questions can be answered, at least in part, with the LQG control model—a technique developed for the control and analysis of physical systems. Most of the developmental work and applications to date have been in the engineering profession. Its use for economic problems is suggested by Athans (1972) who summarizes the method in tutorial fashion. Recent applications by Oudet (1976) and Walsh and Cruz (1976) have developed variations on the LQG method for analysis of macroeconomic systems. The procedure and an analysis of the LQG technique are presented in detail in Section 3.[1] Three strong points recommend its use in studying and making decisions for natural resource problems. First, it is a dynamic control model. Thus, the necessary and sufficient

[1]*Infra,* p. 12.

conditions for its optimality are analogous in principle to the conditions needed to obtain the optimal solution to the actual resource problem. Second, the LQG method explicitly includes many forms of stochastic behavior, and an analytical result (as opposed to numerical or approximate result) can be obtained for many types of stochastic behavior. In addition, the costs of uncertainty as it relates to the objectives of resource management can be evaluated. Third, the solution to the LQG problem is analytical under fairly plausible assumptions. This provides for model experimentation and analysis at a low cost in computer time.

Besides furnishing insights on the effect of uncertainty on problems in the management of natural resources, this study is concerned also with appraising the flexibility and applicability of the LQG technique to a natural resource problem on a microlevel. Since the LQG approach is a comparatively new technique in economic analysis, its performance must be evaluated in an empirical setting. The LQG methodology is set out and examined in Section 3 below and then applied in later sections.

2.6. Summary

This section has derived and interpreted the basic properties of an optimal deterministic natural resource model. The deterministic assumptions do not hold for empirical resource models. When the stochastic nature of a natural resource problem is recognized, the decision–maker is usually faced with a dual problem of optimization with respect to the objective function and a design–of–experiments component. Solutions to this dual optimization problem usually must be computed numerically. This, in turn, is compounded by the "curse of dimensionality" which is a limiting factor for deterministic problems as well as stochastic problems. These two factors force the use of approximation techniques which should be so selected that they are practical and closely resemble the actual problem. The LQG model appears to possess these properties and to be useful for management of natural resource production. To pursue this hypothesis further, the LQG technique is examined more closely in the following section.

3. AN APPROXIMATE STOCHASTIC CONTROL MODEL

Section 3 analyzes the steps and procedures of the LQG method. In 3.1 the LQG optimization philosophy and the analytical solution to the LQG control problem are discussed. Section 3.2 examines the implications of relaxing some of the assumptions of the LQG model that provide for the analytical solution. Approximate solutions to these problems are suggested. Section 3.3 covers one of the more important aspects of the LQG method, particularly as compared with deterministic models—evaluation of the costs of uncertainty.

3.1. The LQG Model and Optimization Philosophy

The LQG optimization technique is a three–step procedure. This section examines each step closely to indicate both the computational steps required and the nature of the approximate optimization problem. The LQG optimization philosophy which briefly is to keep the actual evolution of a system "close" to a predetermined path is analyzed most clearly when discussed simultaneously with exposition of the steps.

3.1.1. Step One: Specification of Target Trajectories

The first step of the LQG procedure is to obtain a set of target levels for the state and control vectors. These targets become the ideal levels that the decision–maker has for the system; any deviation from them results in a cost or loss. The method suggested by Athans (1972) is to assume that algebraic description of the problem is deterministic. Under that assumption, the deterministic control problem is solved to give a time trajectory of targets for the states and controls to attain. Let those respective target values for the state and control variables be denoted by x_t^* and u_t^*. The optimization criterion of the LQG method, as developed in the following two steps, is to keep the real time evolution of the system as close as possible to those targets.

The intent of this first (prior) optimization problem is to define the approximate area where the state and control vectors will be. By assuming that the neighborhood of actual system deviation is small, the modeler can more confidently construct mathematically simplified descriptions of objectives and system dynamics. This confidence is inspired, in great part, by Taylor's theorem, of which explicit use is made shortly. With this prior knowledge, the modeler need not construct objective and dynamics functions in algebra sufficiently complex to cover any possible set of values the system may take on.

Two remarks should be made here. The first is that there is little likelihood that the deterministic solution will be identical to the optimal active learning stochastic solution of the actual problem. That is regrettable but unavoidable. Given that fact, the decision–maker would like to form an idea of how suboptimal the deterministic solution is. An intuitive way of gaining such an insight is to consider that the approximate model has random parameters in a neighborhood of the targets and examines whether an active learning strategy would be beneficial. This topic receives more attention in Section 3.1.3. The second remark concerns the solution of the deterministic problem. Without any additional assumptions on the algebraic form of (3) or (4), obtaining a deterministic solution via dynamic programming may be computationally infeasible if there are more than four or five state variables and five or six time periods. Thus, just to obtain a set of target values, (3) and (4) may have to be converted into an algebraic form that allows for a solution that is computationally feasible.

3.1.2. Step Two: Design, Estimation, and Problem Solution

Functional Form of the LQG Model

Given that the trajectory of target levels has been established and the optimization philosophy is to keep the deviations of the controls and states near their target levels, this section seeks a description of how these deviations evolve over time and a more precise definition and justification of *close*. Since one of the objectives of the approximation scheme is to yield a computationally convenient model, a linear model of variable deviations is developed. The linearizations are obtained via a Taylor–series expansion about the precomputed target levels. The target responses are related by

$$x_{t+1}^* = F_t \ (x_t^*, \ u_t^*) \tag{15}$$

$$z_t^* = \ell_t \ (x_t^*) \tag{16}$$

where (15) is the system dynamics used in deterministic optimization of Step One, and (16) is the output relationship that indicates what observations, z_t, will result given the ideal state, x_t^*, and how the observation process, $\ell(\cdot)$, transforms x_t^* into z_t^*. Upon expanding F and ℓ about x_t^* and u_t^* and defining a new set of variables

$$\delta x_t = x_t^* - x_t \qquad \delta u_t = u_t^* - u_t \qquad \delta z_t = z_t^* - z_t,$$

one can write

$$F_t\ (x_t,\ u_t) = F_t\ (x_t^*,\ u_t^*) + \frac{\partial F_t}{\partial x_t}\Big|_* \delta x_t + \frac{\partial F_t}{\partial u_t}\Big|_* \delta u_t + \alpha_t^*\ (\delta x_t,\ \delta u_t) \tag{17}$$

$$\ell\ (x,\ t) = \ell_t\ (x^*) + \frac{\partial \ell_t}{\partial x_t}\Big|_* \delta x_t + \gamma_t^*\ (\delta x_t) \tag{18}$$

where $|_*$ implies that the partial derivatives are evaluated at the optimal values of the arguments. The α^* and γ^* denote the higher order terms of the Taylor–series expansion. One may generalize and then write:

$$\delta x_{t+1} = A_t\ \delta x_t + B_t\ \delta u_t + \alpha_t\ (\delta x_t,\ \delta u_t) \tag{19}$$

$$\delta z_t = C_t\ \delta x_t + \gamma_t\ (\delta x_t) \tag{20}$$

where the elements of A_t are defined as

$$A_t \equiv \frac{\partial F_t}{\partial x_t}\Big|_* \equiv \frac{\partial F_t}{\partial x_t}\Big|_{x_t^*}^{u_t^*}, \tag{21}$$

an n x n time–varying matrix obtained by evaluating the elements of the Jacobian $\partial F/\partial x$ at the precomputed known time sequences of x_t^* and u_t^*. Similarly,

$$B_t \equiv \frac{\partial F_t}{\partial u_t}\Big|_* \equiv \frac{\partial F_t}{\partial u_t}\Big|_{x_t^*}^{u_t^*} \tag{22}$$

$$C_t \equiv \frac{\partial \ell_t}{\partial x_t}\Big|_* \equiv \frac{\partial \ell_t}{\partial x_t}\Big|_{x_t^*} \tag{23}$$

where A_t, B_t, and C_t are deterministic. Relations (19) and (20) are exact. In computing the solution to the LQG problem, the higher order terms of the Taylor–series expansion are represented by normally distributed random variables with mean zero. Thus, the approximate stochastic model of (4) and (14) is written as

$$\delta x_{t+1} = A_t\ \delta x_t + B_t\ \delta u_t + e_t \tag{24}$$

$$\delta z_t = C_t\ \delta x_t + v_t \tag{25}$$

where $e_t \sim N\ (0, \Omega_t)$ and $v_t \sim N\ (0, \theta_t)$. In (24)–(27), δx_t and e_t are n x 1, δz_t and v_t are m x 1, δu_t is r x 1, A_t is n x n, B_t is n x r, and C_t is m x n. The estimation of Ω_t and θ_t is particularly difficult if (4) and (14) are not linear. Athans (1972) gives no precise formulas for estimating Ω_t and θ_t, but he suggests that Ω_t and θ_t should be designed to reflect the level of importance exerted by the higher order terms of the Taylor series. Thus, the more nonlinear (4) and (14), the larger Ω_t and θ_t would tend to be. The design of Ω_t and θ_t may reflect, to a large degree, the judgment of the designer. When the filter for estimating δx_t is more fully developed, additional insights arise to guide in the interpretation and estimation of Ω_t and θ_t. In one respect the "size" of Ω_t communicates the level of accuracy to the model of the approximate growth dynamics, and θ_t communicates the level of precision of the observation procedure.

Derivation of the objective function for the LQG model is comparatively less structured than the method for obtaining the system dynamics. The form of the approximate objective function is quadratic in δx_t and δu_t, *i.e.*,

$$J_o = \sum_{t=0}^{T-1} (\delta x_t'\ K_t\ \delta x_t + \delta u_t'\ R_t\ \delta u_t) + \delta x_T'\ K_T\ \delta x_T \tag{26}$$

where K_t and R_t are respectively n x n and r x r, and both are positive semidefinite (PSD). The use of a quadratic form has both practical and theoretical justification. On a pragmatic basis, as shown shortly, minimizing the expected value of J_o subject to (24) and (25) yields an analytical solution that can be computed rapidly. On a theoretical basis, Athans (1972) justifies the quadratic form as a means of keeping $\alpha(\cdot)$ and $\gamma(\cdot)$ small which increases the validity of the linear system dynamics. The use of quadratic preferences for economic decision making has been argued by Theil (1964). Some of the more persuasive economic arguments for quadratic preferences are that most economic choices are characterized by diminishing marginal utility and a changing marginal rate of substitution, both qualities exhibited by the quadratic form. In a test of the robustness of the quadratic form, Zellner and Geisel (1968) show that control solutions derived under the assumption of quadratic preferences are close to solutions for the correct specification of the objective function as long as the underlying or true specification is symmetric. The symmetry property, of course, implies that overshooting a goal brings the same loss as does undershooting. This is probably a rare problem since the actual performance of a system is usually less than a theoretical optimal performance. To avoid overshooting targets, Chow (1975) suggests setting the targets slightly beyond what the system can reasonably be expected to achieve. In the context of forestry, some uses of the forest are optimized at less than maximum volumes so that a loss naturally accrues to overshooting a target.

The matrices, R_t and K_t, indicate the relative weight attached to control and state deviations, respectively. If deviations in control actions, u_t, are more costly than deviations in the states, x_t, then the values in R_t will be larger than those in K_t. Athans (1972) discusses one method for selecting appropriate values for R_t and K_t in terms of the original deterministic objectives for some problems. Further guides for selecting the elements can be obtained once the analytical solution to the LQG problem is derived.

The quadratic objective functional (26) explicitly defines the LQG optimizing definition of *close*. What it says is that the system shall be judged to be performing optimally if δx_t and δu_t are both null. Any other value for either vector results in a cost or a loss that increases more than proportionally to the size of the deviation.

Estimating the State Vector

A cursory examination of (24) and (25) shows that stochastic behavior is limited to the two error terms, e_t and v_t, and the initial state estimate, $\delta x_{0|0}$. The assumption that A_t, B_t, C_t, K_t, and R_t are deterministic is restrictive but allows for an analytical solution. The implications for relaxing some of the LQG assumptions are discussed in 3.2. This section addresses the following problem: Given that the objective function is concerned with controlling δx_t and δu_t and that δx_t is *not* assumed to be perfectly observed, what is the best way of estimating δx_t, given all δu_i, $i < t$, all δz_t, $i \leqslant t$, the system model (24) and (27), and no other information? The estimator desired is the one derived in the minimization of the expected value of J_o.

The optimal estimator for the problem posed above is the Kalman filter (Kalman, 1960). The proof of optimality is given in several sources, one being Aoki (1967). The optimality of this estimator for the above problem is due to the separation theorem of Joseph and Tou (1961). The separation theorem states that, for minimizing the expected value of J_o, estimation of the state vector is separate from determination of the optimal controls, given the state vector estimate. Since the Kalman filter is the optimal estimator, this section presents and interprets the recursive equations that generate the optimal estimate.

In addition to the assumptions that Ω_t and θ_t are known, it is assumed that the two error terms are uncorrelated with each other for all t and uncorrelated with the initial estimate of the state deviations. Further, each covariance process is assumed serially independent of itself, *e.g.*, θ_t is independent of θ_{t+i}, $i \neq 0$.[1] Given these assumptions and the others in this section, the Kalman filter can be visualized as a two–step procedure: a prediction stage and an updating stage. Because of the two–step nature of the estimator, it is necessary to introduce some new notation. Specifically, the state variables are given two subscripts such that $\delta x_{i|j}$ is the estimate of δx_i, given all of the observations δz up through and including δz_j.[2] The covariance of $\delta x_{i|j}$ is written $P_{i|j}$, such that $P_{i|j}$ is the covariance of δx_i, given all of the data through period j. Given the normality assumption on e_t and v_t and assuming that $\delta x_{0|0}$ is normally distributed with covariance $P_{0|0}$, it is shown in Aoki (1967) that all of the estimates of δx_t are also normally distributed and that $\delta x_{t|t}$ is a sufficient statistic.

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[1]If the error terms are serially correlated or cross–correlated with each other, the system can be augmented, and results similar to those presented can be derived as discussed in Meditch (1969).

[2]In the optimal solution, δz_t determines δu_t so that, by knowing δz_t, δu_t is also known.

Beginning at the end of period t, the predict cycle begins with predicting δx_{t+1} by

$$\delta x_{t+1|t} = A_t \, \delta x_{t|t} + B_t \, \delta u_t \tag{27}$$

with covariance

$$P_{t+1|t} = A_t \, P_{t|t} \, A_t' + \Omega_t \tag{28}$$

Simply, (27) is a linear combination of $\delta x_{t|t}$ with a constant, $B_t \, \delta u_t$, added so that the covariance (28) follows naturally. The second step commences at the beginning of period t + 1, when a sample is taken so that δz_{t+1} becomes available. A residual value, η_{t+1}, is computed as

$$\eta_{t+1} = \delta z_{t+1} - C_{t+1} \, \delta x_{t+1|t} \tag{29}$$

which is easily interpreted as the difference between what would be expected to be observed, given the predict estimate, and what is actually observed, δz_{t+1}. If $\eta_{t+1} = 0$, then, intuitively, the new sample data agree with what is predicted, and no change is needed to update $\delta x_{t+1|t}$. Rarely is η_{t+1} null. Thus, in the second step (update cycle), the information gained by sampling must be combined optimally with $\delta x_{t+1|t}$ to yield the estimate of $\delta x_{t+1|t+1}$. To do this, it is first necessary to compute $P_{t+1|t+1}$ as

$$P_{t+1|t+1} = P_{t+1|t} - P_{t+1|t}' \, C_{t+1}' \, (C_{t+1} \, P_{t+1|t} \, C_{t+1}' + \theta_{t+1})^{-1} \, C_{t+1} \, P_{t+1|t}. \tag{30}$$

The optimal estimate is given as

$$\delta x_{t+1|t+1} = \delta x_{t+1|t} - P_{t+1|t+1} \, C_{t+1}' \, \theta_{t+1}^{-1} \, \eta_{t+1}. \tag{31}$$

The matrix, $P_{t+1|t+1} \, C_{t+1}' \, \theta_{t+1}^{-1}$, is referred to as the Kalman gain matrix. It shows how the residual vector should be weighted in adding it to $\delta z_{t+1|t}$ which has the identity as an implicit weighting matrix in (31). If the observation error is large relative to the predict error, $P_{t+1|t}$, then from (30) it is clear that the second term on the right–hand side of (30) will become quite small, and $P_{t+1|t+1} \approx P_{t+1|t}$. Additionally, in (31) a large θ_{t+1} causes θ_{t+1}^{-1} to be very small so that little significance is attached to the residual vector as is intuitively expected. If the observer is relatively inaccurate, then little reliance should be placed on the observation as a source of information about $\delta x_{t+1|t+1}$. Just the opposite result obtains if θ_{t+1} is small relative to $P_{t+1|t}$.

Two remarks are in order here. First, observe that (28) and (30) are based on information that is known before the actual control problem begins. Thus, the $P_{i|j}$ is precomputable and unaffected by the control actions. This is a result of the separation theorem. The impact of the relative sizes of Ω_t and θ_t on the estimates provides an additional guideline in estimating these covariances. If the modeler knows that the observation process is more inaccurate than the predict process, then this should be reflected in the construction and estimation of Ω_t and θ_t and will result in θ_t being *larger* than Ω_t.

Derivation of the Optimal Controls

The separation theorem and the solution to the LQG problem are well known in the control literature. Even so, the solution to the control segment of the problem is derived here in some detail to underscore an assumption that is troublesome for economic problems in which some or all of the control and state variables have limited feasibility regions. Backward recursion, as employed in dynamic programming, is used to derive the optimal solution.

The *T*th period problem is to minimize the expected value of J_{T-1} given $\delta x_{T-1|T-1}$ where

$$J_{T-1} = \delta x'_{T-1} K_T \delta x_{T-1} + \delta u'_{T-1} R_{T-1} \delta u_{T-1} + \delta x'_T K_T \delta x_T. \tag{32}$$

Anticipating the generalization to multiple periods, let $H_T = K_T$ so that, substituting into (32) for δx_T from (24) and gathering like terms and employing the expectation operator, (32) can be rewritten as

$$\begin{aligned} E(J_{T-1}) = E \Big\{ & [\delta x'_{T-1} (K_{T-1} + A'_{T-1} H_T A_{T-1}) \delta x_{T-1}] \\ & + 2\delta u'_{T-1} [B'_{T-1} H_T A_{T-1}] \delta x_{T-1} + \delta u'_{T-1} \\ & [B'_{T-1} H_T B_{T-1} + R_{T-1}] \delta u_{T-1} \Big\} + \operatorname{tr} H_T \Omega_{T-1}. \end{aligned} \tag{33}$$

To obtain the optimal control, the gradient of the expected value of (33) with respect to δu_t is set equal to zero giving:

$$\delta u_{T-1} = -G_{T-1} \delta x_{T-1|T-1}$$

where

$$G_{T-1} = (B'_{T-1} H_T B_{T-1} + R_{T-1})^{-1} B'_{T-1} H_T A_{T-1}.$$

It can readily be shown, as in Meditch (1969), that the solution generalizes to

$$H_t = K_t + A'_t H_{t+1} A_t - A_t H_{t+1} B_t G_t, \quad H_T = K_T \qquad t = 0, \ldots, T-1 \tag{34}$$

$$G_t = (B'_t H_{t+1} B_t + R_t)^{-1} B'_t H_{t+1} A_t \qquad t = 0, \ldots, T-1 \tag{35}$$

$$\delta u_t = -G_t \delta x_{t|t} \qquad t = 0, \ldots, T-1. \tag{36}$$

Two remarks can be made about the solution of (34)–(36). First, the linear quadratic minimization problem is a control problem. Thus, the necessary conditions that (36) satisfies are analogous to those satisfied for production levels in the deterministic problem. Hence, the LQG model must weigh the current cost of control deviations against the user cost of deviating from the desired level of resource stocks in the future. The second remark is that the effect on δu_t of the relative sizes of K_t and R_t can be analyzed in

(36). As R_t becomes *larger* relative to K_{t+1}, $(B'_t\ H_{t+1}\ B_t + R_t)^{-1}$ becomes *smaller.* Hence, less change is made in the control vector for a given $\delta x_{t|t}$. The opposite effect, however, holds if K_{t+1}, is large relative to R_t. Then the controls become increasingly responsive to $\delta x_{t|t}$. These effects can help guide the choice of the relative weights of K_t to R_t in the design procedure of the objective function in step two.

The expected value of J_t, given $\delta x_{0|0}$, δz_1, ..., δz_t, is:

$$E\ (J_t \mid \cdot) = \delta x'_{t|t}\ H_t\ \delta x_{t|t} + \text{tr}\ H_t\ P_{t|t} + \sum_{i=t}^{T-1} \text{tr}\ H_{i+1}\ \Omega_i + \sum_{i=t}^{T-1} \text{tr}\ A'_i\ H_{i+1}\ B_i\ G_i\ P_{i|i}. \tag{37}$$

Since the last three terms of (37) are constants, they can be computed before the system starts to evolve in real time. These terms have an interpretation as costs of uncertainty and are discussed in more detail in 3.3.1.

Sufficient conditions for a minimum are fulfilled if all of the R_t and K_t are PSD. This can be verified by examining the Hessian of (33) with respect to δu_t and observing that H_t will always be PSD given that R_t and K_t are PSD. This latter aspect is difficult to ascertain from (33) but is clearly illustrated in Chow (1975).

3.1.3. Step Three: The Real Time Solution

The prior two steps occur before the system to be controlled begins to evolve. Once the system begins evolving, the control process is quite direct. At t = 0, the δu_0 is computed given $\delta x_{0|0}$ via (36). At the end of the initial period, $\delta x_{1|0}$ is computed using (27). An observation is then taken to generate δz_1. Given δz_1, $\delta x_{1|1}$ is computed using (31). Then the whole cycle can be repeated until the end of the planning horizon. Clearly, given δu_t, u_t can be computed directly.

To analyze the system for various properties or policies or to validate the model itself, experimental runs are typical. Such require, in many instances, simulating a series of δz_t, t = 1, ..., T. This can be done by using historically observed values or by generating normally distributed random variables given the covariances of $\delta x_{0|0}$, e_t, and v_t.

3.2. Implications of Relaxing the LQG Assumptions and Suboptimal Controls

Section 2.4.2 argued that the management of natural resource production is a stochastic control problem and that the optimal solution can be obtained only when active learning strategies are employed. The solution to the LQG problem, derived in 3.1.2 is optimal without employing active learning strategies. Optimality of the solution results because all of the model parameters except e_t and v_t are assumed to be known constants. When those assumptions are relaxed, as realism suggests, then active learning strategies become a relevant consideration. An additional problem that frequently arises in control models of natural resources is keeping stock and production variables feasible. Because of the unconstrained nature of the solution, it cannot be assumed that use of the LQG model will guarantee feasible controls.

The active learning aspect of control and possible approximations with uncertain linear dynamic coefficients are discussed in 3.2.1.[1] The control constraint problem is pursued in Section 3.2.2.

3.2.1. The LQG Model With Stochastic Coefficients in the System Dynamics

This section examines the implications of relaxing the assumption that A_t and B_t are known with certainty. It determines the impact of assuming that A_t and B_t are fixed when they are actually random, and it goes on to discuss when that assumption results in a nearly optimal solution. That is, it considers when an active learning solution is clearly required and when it can be closely approximated by a nonactive learning solution.

To develop the above ideas, some of the problems involved can be best illustrated by assuming that A_t and B_t are random but that their joint density function is completely known for all t at t = 0 and that C_t, Ω_t, and θ_t are still assumed known. In such a case an active learning strategy is not called for since nothing can be done to change the level of uncertainty in the A_t or B_t. If it is assumed that the state vector is perfectly observed, then this problem has an analytical solution as developed in Aoki (1967). The solution, a modified version of (34)–(36), is referred to as a pure stochastic solution. In the following discussion the solution when A_t and B_t are assumed known is called the certainty–equivalent solution. Developing the stochastic problem in a precise algebraic form depends on obtaining the expected value of the product of three matrices, XYZ, where X and Z are random. That can be accomplished by defining a stacking operator $\phi(M)$, where M is any i x j random matrix as $\phi'(M) = [m^{1'}, m^{2'}, ..., m^{j'}]$ where m^k is the *k*th column of M. Thus, $\phi(M)$ is an ij x 1 vector. Then Γ_{XZ} can be defined as the covariance of the $\phi(X)$ and $\phi(Z)$ vectors. Using the star product defined by MacRae (1971),[2] the expected value of XYZ is

$$E\,(XYZ) = Y * \Gamma_{XZ} + \overline{XYZ} \tag{38}$$

where a bar denotes the mean. Writing E(XYZ) as $\overline{XYZ}$, the optimal solution to the LQG problem, assuming perfect observation of x_t, is:

$$H_t = K_t + \overline{A_t' H_{t+1} A_t} - \overline{A_t' H_{t+1} B_t G_t} \qquad H_T = K_T \tag{39}$$

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[1]Active learning strategies for uncertain preferences are not well developed in the control literature. Also, the assumption that the coefficients C_t are known is largely in agreement with reality.

[2]The star product of an m x n matrix A and an mp x nq matrix B is a p x q matrix C such that

$$C = A * B = \sum_{i,j}\sum a_{ij} B_{ij}$$

where a_{ij} is the *ij*th element of A, and B_{ij} is the *ij*th (p x q) submatrix of B.

$$G_t = \overline{(B_t' H_{t+1} B_t} + R_t)^{-1} \overline{B_t' H_{t+1} A_t} \tag{40}$$

$$\delta u_t = -G_t \ \delta x_t . \tag{41}$$

In empirical applications the stochastic solution is frequently used as an approximate solution to the active learning problem. The stochastic solution can be modified to be an open–loop feedback solution that employs passive learning. That strategy assumes that the estimated density of A_t and B_t will remain constant for the planning horizon, whereas, in fact, it will be updated. A solution is calculated on that assumption, and the initial period control action is taken. At the end of the time period, new data are available in the form of an additional observation on δx_t and δu_t so that the estimates of A_t and B_t can be sequentially updated. A new solution is computed, once again assuming that the density of all A_t and B_t are known, and the procedure is continued in each time period. Thus, the open–loop feedback solution uses the stochastic solution and sequential updating.

The certainty–equivalent and open–loop feedback solutions can be viewed as approximations to the truly optimal active learning solution when A_t and B_t are assumed to be unknown constants. Prescott (1972) investigated the conditions under which one of these solutions is a satisfactory approximation for an active learning solution assuming δx_t is observed perfectly. He used a simple scalar model for his numerical problem and determined that the value of experimentation, *i.e.*, active learning, increases with the number of time periods in the problem and the degree of uncertainty in the unknown coefficient. His results indicate that the certainty–equivalent solution is an acceptable approximation to the active learning solution when the prior estimate for the unknown parameter is at least four times its standard error in absolute value. The stochastic solution provides a reasonable approximation of the active learning solution for determining initial period controls when the parameter estimate is at least twice its standard error. When a coefficient is less than twice its standard error, experimentation becomes an important consideration. Since the linear dynamics used in the study by Prescott is quite simple, his results cannot be categorically generalized to models of greater linear complexity and dimension. Unfortunately, no better information or guidelines appear to exist for stipulating when active learning can be approximated closely by approximate techniques. More robust guidelines are clearly called for in future research efforts.

In discussing when an active learning strategy can be accurately approximated by a certainty–equivalent solution, it should be borne in mind that the underlying stochastic control problem represented by (3) and (4) may require an active learning solution even if the LQG system's coefficients are statistically significant. Thus, the modeler must make the assumption that, if the approximate LQG system dynamics does not require active learning strategies, then the open–loop feedback or certainty–equivalent solutions are a reasonable approach to the solution. Such an assumption is similar to the LQG assumption that the optimal path for the system is close to the ideal trajectory yielded by the deterministic optimal solution of Step One of the LQG technique. If the statistical significance suggests that an active learning solution is appropriate, the feasibility of employing approximate solutions that utilize some experimentation should be considered. For example, MacRae (1972) and Popović (1972) give explicit expositions of approximate solutions that incorporate learning. Additionally, most of the empirical solutions using active learning strategies assume that there is a straightforward relationship between the parameters in the current and future periods. Since the parameters are derived via Taylor's theorem in LQG models and the relationship between parameters over time periods may be quite complex, the relevance of Prescott's findings to LQG models is further questioned.[1]

When the assumption is relaxed that δx_t is observed perfectly, the control problem becomes even more difficult with A_t and B_t uncertain. Consider, first, the case where A_t and B_t are random but their density is completely known at $t = 0$, for all t. The Kalman filter is no longer optimal for several reasons. One of the clearest is that A_t δx_t is the product of two random matrices. This implies that the variance of $\delta x_{t+1|t}$ is a function of the levels of both the mean of A_t and $\delta x_{t|t}$. Recall from (28) and (30), the covariance propagation matrices, that all $P_{t|t}$ are independent of the level of δx_t, for all t. From (37), it is also clear that, when the $P_{t|t}$ becomes a function of the level of the state vector, the costs of uncertainty terms involving $P_{t|t}$ are no longer constants. The expressions involving $P_{t|t}$ become part of the control problem, and the separation principle no longer applies. Solving the integrated control estimation problem requires numerical methods, so the convenience of an analytical solution is lost. If the assumption that the density of A_t and B_t is known at $t = 0$ is relaxed, solving the LQG problem is all the more difficult. Optimality requires not only an active learning solution but simultaneous solution with the observation problem.

Approximations must be made in solving empirical problems. The LQG model is such an approximation since it assumes away all uncertainty except in e_t, v_t, and $\delta x_{0|0}$. In the empirical applications of Section 5, a stochastic solution using (39)–(41) is computed under the assumption that the separation theorem between control and estimation still holds. The Kalman equations are used to derive an estimate of the inaccessible state vector. The implications of using the Kalman filter for random A_t and B_t vary with the particular densities of A_t and B_t. Section 4 discusses the implications of that approximation for the empirical application.

3.2.2. Feasibility Constraints

The unconstrained optimization of (32) implies that δx_t and δu_t may take on any real value. In the physical example of a forest, it is clear that such an assumption is not valid since it is impossible to harvest nonexistent volumes. Feasibility constraints are quite common in many natural resource problems. This section analyzes the feasibility problem and an approximate solution.

In the typical dynamic programming problem, the curse of dimensionality is often confronted when there are more than four or five state variables. A remarkable property of the LQG method is that no such constraint materializes. The actual solution is obtained by generating (34) and (35) by backward recursion from T and storing them on disk files. The controls are computed moving forward in time as $\delta x_{t|t}$ becomes available. Assume for the moment that the LQG problem is deterministic and that the recursive relations

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[1] Clearly, if something has been learned about future parameters in the LQG, the deterministic model in Step One can be updated; hence, a new set of targets would be generated. Given this link, the optimal level of experimentation using the LQG model is an extraordinarily complex problem.

(34) and (35) have been computed.[1] It is then possible to generate $\delta u_0, \ldots, \delta u_{T-1}$. If any of those controls violate *a priori* constraints, the solution is not feasible, although—given the proper convexity assumptions on K and R—the expected cost of this infeasible solution is a lower bound on the cost of the optimal feasible solution. To compute the truly optimal constrained solution by dynamic programming would be extraordinarily costly for most problems with today's computers. Given the quadratic–linear nature of the LQG problem, however, it can be converted to a quadratic programming problem. For a problem with large dimensions, this approach also can become computationally intractable.

The constrained problem becomes more complex when the deterministic assumption is relaxed. The problem also can be solved by quadratic programming, although it is then necessary to solve a quadratic program in which some of the parameters are random. Again, dimensionality would be a significant problem.

Given these difficulties, it can be argued that a solution to a stochastic tracking problem for all possible random events requires that an approximation be used. One approach is the use of penalty functions. One variation of that approach is to assign a large weight in the objective function to variables that tend to infeasible values so that the solution to the problem will avoid large violations of feasibility. From an economic standpoint, such a scheme may be difficult to rationalize. Additionally, in stochastic problems the use of penalty functions may require numerous iterations to determine which variables need to be penalized. Some aspects of penalty functions are discussed in Sage (1968).

An alternative method (employed in Section 5) is to restrict changes in variables only to those variables that do not lie on a feasibility boundary. For example, suppose that no harvests are scheduled in a given stand during a particular period. The target harvest level is zero so a negative $\delta u_{t,i}$ is infeasible. Thus, this control is on its feasibility boundary. The problem is avoided by designing B_t such that those components of δu_t that lie on a feasibility boundary cannot be changed as discussed in 4.2.2. The restriction can be justified on the grounds that, most of the time or by design, the actual system performance will result in somewhat less than all the target values so that the control variables will be less than their target levels. In addition, in the neighborhood of the optimum, radical shifts in variable levels would not be expected. Thus, variables lying on a feasibility boundary, particularly those at a zero level, would probably remain at that boundary, especially if the target levels are set such that the actual system performance undershoots the target levels.

The above restriction does not entirely rule out the possibility of infeasible controls. If there are large variations in $\delta x_{t|t}$, then the possibility of infeasible controls certainly exists. When such large disturbances occur, the use of a tracking model may be overshadowed by the need for a new set of targets to be tracked. In the application, infeasible unconstrained variables are rare, occurring only with some controls. When they do occur, the control is set at its closest feasible level, similar to the approach in Kim, Goreux, and Kendrick (1975). This procedure results in a suboptimal feasible solution. How suboptimal the controls and objective function value will be in comparison with

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[1]That is, e_t and v_t are degenerate random vectors.

other approximate techniques remains to be established. This question requires further research, with important implications for the use of control models in solving economic problems.

3.3. Evaluating and Interpreting the Costs of Uncertainty

As discussed in Section 2.4.1, the introduction of uncertainty into a problem broadens the set of decisions that a manager must make. Part of this expansion of the decision set involves the consideration of using active learning strategies. Another aspect of uncertainty is evaluating the costs of the several sources of uncertainty. On the basis of these comparative costs, the decision–maker can decide where research or sampling efforts (passive learning) should be directed toward lessening uncertainty. In the LQG model it is possible to evaluate the various costs of uncertainty. Since two versions of the LQG model have been analyzed, *i.e.*, one with known A_t and B_t and one assuming them to be uncertain, the two models are discussed separately. The costs of the sources of uncertainty for the certainty equivalent model in 3.3.1 are examined, and the costs of uncertainty engendered by uncertain A_t and B_t are analyzed in 3.3.2.

3.3.1. Uncertainty in the Certainty–Equivalent Model

Under the certainty–equivalent (CE) assumptions, uncertainty is represented in the model by $\delta x_{0|0}$, the randomness of e_t, and the sampling error v_t. The costs of these sources of uncertainty are given in (37) below which is rewritten here for convenience.

$$E\ (J_t \mid \cdot) = \delta x'_{t|t}\ H_t\ \delta x_{t|t} + \operatorname{tr} H_T\ P_{t|t} + \sum_{i=t}^{T-1} \operatorname{tr} H_{i+1}\ \Omega_i + \sum_{i=t}^{T-1} \operatorname{tr} A'_i\ H_{i+1}\ B_i\ G_i\ P_{i|i}.$$

All of the terms in (37) are scalars, and H_t and $A_i\ H_{i+1}\ B_i\ G_i$ are symmetric PSD so that all four terms are always nonnegative. Each of the above terms has an economic meaning. The first term represents the total cost, present and future, of the estimate of x_t deviating from the desired state, x^*. Recall that the maximum principle yields results identical to those with dynamic programming so that the gradient of the first term with respect to $\delta x_{t|t}$ would yield an estimate of the user cost of deviating from the goals by an increment to $\delta x_{t|t}$, *i.e.*, the estimated marginal cost of welfare over the rest of the planning period of an increase in the deviation from current targets.

The last three terms of (37) are independent of the level of $\delta x_{t|t}$. The second term quantifies the cost increase due to current uncertainty in δx_t. As Athans (1972) states, this term couples the effects of the control cost functional quantified by H_t with the current uncertainty of estimation, $P_{t|t}$. A similar interpretation can be given to the fourth term. It represents the current cost of future actions arising because of uncertainty in the current and future estimates of $\delta x_{t|t}$. The relation of the $P_{t|t}$ to the period–by–period control cost functional is given by $A_i\ H_{i+1}\ B_i\ G_i$. The third term in (37) represents the current cost of future system dynamics uncertainty over the entire planning horizon. In all three of these terms, the larger the respective covariance matrices, the greater the cost. Hence, the terms of (37) indicate to the decision–maker the cost of various types of uncertainty, and research efforts can thus be allocated in an appropriate way. In determining the allocation of research efforts, the prospective return to research investment must also be considered; however, this is not a parameter of the control problem. Thus, the optimal rate of research is not given as a result of the LQG model. For example, the LQG model does not indicate the optimal rate of sampling since it does not include sampling costs. Rausser and Howitt (1975) examined a control problem that included sampling costs and found that an analytical solution exists only for a specific case.

The analysis in this section can be extended by taking the partial derivatives of the three constant terms to obtain marginal costs of uncertainty. Many useful trace derivatives are given in Athans (1968).

Given that H_t is a function of the K_i, R_i $(i \geqslant t)$, it is quickly deduced that the costs of uncertainty are related directly to the objective function. That is, under varying policies, as represented by K_t and R_t, the costs of various sources of uncertainty will change as K_t and R_t change. This fact gives the decision–maker a method of analyzing the costs of uncertainty under a variety of policy regimes to determine the areas of research that have the potential for the highest returns from research. From (37), it is clear that these costs of uncertainty are in the same units as K_t and R_t. If the units of K_t and R_t are in dollars, then an unambiguous interpretation can be given to the costs of uncertainty. If, as is more likely in the case of publicly owned resources, the units of R_t and K_t do not have a dollar unit, evaluation of the costs becomes more complex. If the utility index represented by K_t and R_t is derived under the assumption of maximizing expected utility, it would be possible to subtract the cost of decreasing uncertainty from the income or budget and then determine whether the resulting expected utility justifies the cost of the increased information. In any event, expected utility can be increased by a decrease in uncertainty, given a quadratic preference function. Thus, in terms of minimizing disutility, a quadratic loss function is risk averse. When the LQG loss function is in an arbitrary utility unit, no cardinal significance can be attached to the last three terms in (37) since they vary proportionately with a linear transformation of the K_t and R_t. However, the percent of costs, accounted for by various sources of uncertainty, are invariant under a linear transformation of K_t and R_t.

3.3.2. Costs of Uncertainty With Stochastic Linear Coefficients

The analysis in 3.3.1 applies equally to the stochastic model (S); but, under the more general assumptions of S, evaluation of all sources of uncertainty becomes more complicated since H_t is a function of the covariance matrices (Γ) of the various linear coefficients. Evaluation of the costs of uncertainty generated by the presence of the variances and covariances of A_t and B_t is quite complicated; in general, few analytical results can be derived.

Assuming for the moment that δx_t is observed perfectly, Chow (1975) has shown that it is not possible to conclude that uncertainty in the linear–dynamics coefficients will lead to higher expected costs. In terms of the cost matrix, H_t, this implies––letting superscripts denote the stochastic specification of the model––that H_t^{CE} is not necessarily less than H_t^S by a PSD matrix. Thus, to evaluate the cost of uncertain linear coefficients, the specific numerical values must be computed.

One distinction between the CE model and S model is that uncertainty in the A_t and B_t may add to the cost of current state deviations as measured by the first term of (37), $\delta x_{t|t}' H_t \, \delta x_{t|t}$. If $\delta x_{t|t} = 0$, then any cost of uncertainty associated with H_t^S is zero in terms of current deviations from desired goals. Intuitively, this makes sense.

Even if a system exhibits randomness, there is no cost if the system is exactly on target since no control action can be taken to better the state of the system. When a system is considerably off target, however, $\delta x_{t|t}$ is large, and the uncertainty associated with current state deviations will then become increasingly costly, assuming that the random factor adds a positive definite matrix to H_t^S.

The effect of uncertain A_t and B_t is partially manifested also in the latter three terms of the objective function. Clearly, since H_t is affected by the uncertainty of A_t and B_t, the costs of Ω_t and $P_{t|t}$ will most likely change. Thus, the total impact of random A_t and B_t on costs of uncertainty can only be evaluated by numerical computations.

3.4. Summary

This section has briefly examined the problems of obtaining an empirical solution to a stochastic control problem and has suggested an approximate optimization method. The technique, the LQG model, is a three–stage optimization problem embedding one optimization process inside another. The LQG approximation reduces the original problem into a much simpler algebraic form. The simplified form can then be examined to determine whether active learning strategies are appropriate. Even though the LQG model presents the system dynamics in a linear form, once the uncertainty in the linearized coefficients is recognized, the convenience of an analytical solution to the control and state vector estimation problem is lost, and approximate estimators and control algorithms must be used to keep computational costs at a level that allows inexpensive model experimentation. An additional problem that may arise with the LQG technique for economic models is from feasibility constraints. Careful design of the LQG model helps avoid most problems in that area, but some resource problems may not be amenable to the LQG method if the feasibility problem does not permit reasonable approximations. A final attribute of the LQG analysis for this study is that the costs of uncertainty can be evaluated in relation to the decision–maker's objectives, and various sources of uncertainty can be ranked at least in terms of percentage of total costs.

4. MODEL SPECIFICATION AND PARAMETER ESTIMATION

Given the theory in Sections 2 and 3, the study now turns to application of the LQG technique, examining how effectively the LQG technique can be used for timber scheduling under uncertainty on the Stanislaus National Forest. The harvest schedule for the standard component[1] in the Stanislaus National Forest is determined with the Timber Resources Allocation Method (Timber RAM or RAM) which is a linear programming model (Navon, 1971). Use of a Timber RAM model provides the deterministic model needed in Step One of the LQG method.

The empirical analysis is limited to the mixed–conifer timber type which comprises 134,720 acres of the 275,330 acres in the standard component. That limitation is made since this study examines the application of stochastic control to timber management problems and requires extensive model experimentation. The pine and red fir timber types that make up the balance of the standard component in the Stanislaus National Forest are excluded to diminish computing costs due to the dimensionality of the matrices in

[1] Defined as those lands within the Stanislaus National Forest suitable for timber production under methods of intensive management.

(34) and (35). Their inclusion would generate no theoretical difficulties but would make model experimentation and simulation much more costly.

RAM, the deterministic model used in Step One of the LQG method, is described briefly and analyzed in Section 4.1. The growth prediction model used for RAM on the Stanislaus National Forest is analyzed and then approximated into a linear model. The linearized dynamics is interpreted, and its accuracy as a representation of a timber management problem is evaluated. The ensuing sections then detail how the exact parameter estimates are obtained for constructing the objective function, the equations of motion, and the observer relationship.

4.1. Timber Resources Allocation Method

Timber RAM is a linear programming model developed by Navon (1971) for scheduling regeneration harvests and thinnings. It can be specified to incorporate various goals such as maximum present value, maximum sustained yield, or nondeclining flow. The last objective implies that the harvest vectors, u_t, are recursively related such that letting i be a column vector with unit values for each component,

$$i'u_t \leqslant i'u_{t+1} \qquad t = 0, 1, ..., T.$$

The constraints and technical coefficients are specified by the user to represent the particular characteristics of the stands under consideration. Since many national forests do not have a complete road system throughout their standard component, harvests can be restricted in such areas until the dates of anticipated road completion.

The length of the decision period for Timber RAM is a decade, and the planning horizon is 38 decades. The output from Timber RAM can be utilized to determine (1) the quantity of timber to be harvested in regeneration cuts and thinnings and (2) the level of timber stocks for each condition class of timber. The model implicitly assumes that the acres within any condition class are homogeneous, thus avoiding aggregation difficulties. Technological change must be perfectly anticipated and incorporated into the elements of the coefficient matrices.

Timber RAM has been critically reviewed by Chappelle, Mang, and Miley (1976). They observe that, although RAM employs linear programming, the structure of the model and the method for entering data severely restrict the feasibility region of the activity vector. Data are entered in tabular form, and the length of rotation for a stand is given as input. The levels of entry and reentry cuts per acre are entered as parameters for the condition classes. In addition, RAM is a "point" model since it neglects the distribution of areas, ignoring aspects of planning such as the environmental impact that harvesting in one area has on an adjacent area. RAM also does not schedule cultural treatments other than thinning.

Another drawback of RAM according to Chappelle, Mang, and Miley (p. 290) is that "Timber RAM focuses on optimizing timber outputs thereby neglecting other multiple uses of the forest, except insofar as lands dedicated to other uses are subtracted from the resource availabilities entered into the model." Chappelle, Mang, and Miley further suggest that the widespread use of RAM and the ". . . low availability or nonavailability of methods to schedule other forestry outputs encourages misallocation of resources in favor of timber production." Thus, the use of RAM to schedule harvest on a portion of a national forest most likely does not yield a solution that is optimal with respect to the overall management objectives. But, as Chappelle, Mang, and Miley (p. 293) state, "No existing models can provide answers to these broad questions."

An important property of the model (also true for the current models that give long–run plans for large forest tracts) is that the harvest schedule cannot be applied directly to a forest. The reason is that organization of an actual harvest requires on–the–ground judgment by a local forester concerning possible violations of any environmental or ecological constraints. Such constraints can be so critical in nature and so numerous and variable within relatively small regions that the costs of data collection and resulting computer analysis are prohibitive. Classification and stratification of complex natural systems to allow quantification in a model can always be criticized as "too aggregated and unrealistic." The criteria by which the specification should be judged is whether the specification captures the characteristics of the natural system at the same level of aggregation at which policy decisions based on the model are made. Exact specification of a complex system subverts the concept of a model and would lead to "too fine" an information grid for the system (Marschak and Miyasawa, 1968). Thus, the lack of complete problem formulation presents a type of specification error in large–scale models that must be considered in utilizing the output results.

A major point of departure between Timber RAM and the proposed model is that the former is strictly deterministic, and there is no explicit strategy for responding to stochastic events. For example, it is assumed that there is no error in observing the states, a tenuous assumption since the forest inventory is estimated by decennial sampling. In addition, considerable uncertainty exists about projected yields. Thus, it is important to examine the effects and costs of stochastic behavior.

4.2. The LQG Model as a Representation of a Timber–Harvesting Problem

In this section the objective function, system dynamics, and observer relationship of the LQG model are interpreted as an approximate model of the harvest–scheduling problem solved by Timber RAM. Derivation of the A_t and B_t matrices requires a detailed exposition of how timber growth is modeled in RAM; this is presented in 4.2.1. Given this understanding of growth as modeled in RAM, the approximate growth system used in the LQG model is explained in 4.2.2. Because of the way the growth dynamics in the LQG model is structured and estimated, the flexibility of the LQG model is limited. The limitations are discussed in 4.2.3.

4.2.1. Growth Prediction in the Stanislaus National Forest RAM Model

The growth dynamics used in the RAM model to generate the needed targets in Step One of the LQG is discussed and presented in U. S. Forest Service (1974b) for the Stanislaus National Forest. In the growth dynamics, merchantable volume is given as a function of basal area and height. Thus, the growth system consists of three equations: (1) to give height as a function of age,[1] (2) to give BA_{t+1} as a function of BA_t and age, and (3) to give Vol_{t+1} as a function of BA_{t+1} and current stand height. However, since height is a function solely of age, the equation giving height can be eliminated by making volume a function of BA and stand age. The growth system then becomes

[1]Computed as an average weighted by basal area.

$$\mathrm{Vol}_{t+1} = F_{1t}\ (BA_{t+1},\ U_t) \tag{42}$$

$$BA_{t+1} = F_{2t}\ (BA_t,\ U_t) \tag{43}$$

where

Vol = merchantable volume

BA = basal area

and

U_t = timber removal actions.

By excluding the height variable, the dimensionalities of the LQG matrices to represent (42) and (43) are decreased substantially. This provides a large savings in computational costs for the LQG model.

A somewhat striking aspect of (42) and (43) is that $\mathrm{Vol}_{t+1} \neq F\ (\mathrm{Vol}_t)$. Volume is determined by basal area and stand age. Thus, volume for a given area is determined by BA_t for that area and the stand age. Additionally, volume is never directly observed in this system so that sampling is done only for BA_t.[1] A further characteristic of the RAM model is that the timber removal actions are taken at the beginning of the period, and volume and basal area in the next period are predicted given the residual basal area.[2] This can be easily modeled in the LQG as discussed in 4.2.2.

4.2.2. An Approximate Model of RAM

Application of the LQG model to a harvest–scheduling problem on a national forest requires that the objectives of management be specified as a quadratic form of the following type:

$$J = \sum_{t=0}^{T-1} (\delta x_t'\ K_t\ \delta x_t + \delta u_t'\ R_t\ \delta u_t) + \delta x_T'\ K_T\ \delta x_T \tag{44}$$

where δ indicates that a vector is in deviation form, *i.e.*, $\delta x_t = (x_t^* - x_t)$. In this application the vector x_t is 33 x 1, its first 15 components are volumes of merchantable timber in millions of cubic feet, and its last 18 components are in thousands of square feet of basal area. All components denote levels at the beginning of t. The state vector is defined in further detail shortly. The vector δu_t is 20 x 1, and its components are the volume (in millions of cubic feet), removed by thinnings or regeneration harvests at the beginning of t. The K_t are 33 x 33, and the R_t are 20 x 20. These matrices,

[1]For regenerated stands, uniform establishment is assumed which allows average stand age to be known with certainty. To avoid excessive model dimensionality problems with the LQG solution, it is assumed that the age of the wild stands is known with certainty.

[2]Due to the limited data base to estimate (42) and (43), the effects of mortality, weather, and other exogenous effects on the forest are represented by the additive error terms as in (24).

respectively, represent the cost (or loss) of not having the predetermined timber stocks, x_t^*, or the desired harvest levels, u_t^*. Selection of the weighting values for K_t and R_t is discussed in 4.3.1.

The approximate timber growth dynamics is modeled as:

$$\delta x_{t+1} = A_t \, \delta x_t + B_t \, \delta u_t + e_t \tag{45}$$

where

δx_t and e_t = 33 x 1 vectors

A_t = 33 x 33 matrix

and

B_t = 33 x 20 matrix.

To understand (45), observe that, in the system used to simulate growth in RAM, (42) and (43), three kinds of information are given for every acre: basal area, volume, and stand age. Since the linear coefficients in (41) are a function of age, the growth dynamics equation (45) models only basal area growth and volume, given basal area. The basic procedure used to model the growth and volume by (45) is to use two interrelated systems that (1) models growth of wild stands and (2) models the growth of regenerated stands. To understand that system, δx_t is partitioned as:

$$\delta x_t' = \left[\delta V_t', \ \delta BA_t^{1'}, \ \delta BA_t^{2'}\right]$$

where

δV_t = 15 x 1 vector

δBA_t^1 = 3 x 1 vector

and

δBA_t^2 = 15 x 1 vector.

The components of δV_t are volume deviations in given timber classes, and the components of δBA_t^2 are the deviations in basal area in the corresponding timber classes. For example, the first component of δV_t is the deviation in volume of a given class, and the first component of δBA_t^2 is the corresponding deviation in basal area on that acreage. Since it is assumed in the RAM model that no timber can be utilized for postharvest use until a stand is 35 years old, the three components of δBA_t^1 represent the deviations in basal area of the 5–, 15–, and 25–year–old regenerated timber classes. All of the state variables are given at mid–decade as in RAM, and RAM regeneration occurs five years after a regeneration harvest. The LQG period begins and ends at mid–decades.

The mixed–conifer species initially has seven timber classes as listed in Table 1. The first six (M01–M42) represent wild stands, and the seventh (MP) designates recently regenerated. The volume and basal area on sites within M01–M42 are the last six components of δV_t and δBA_t^2, respectively. For example, assume that time t starts at zero; when t = 2, the 11th component of δV_t is the deviation in volume of timber in M10 and the stand is 115 years old. The 11th component of δBA_t^2 is the deviation in basal area in M10. Once an area in one of the M01–M42 classes is regeneration–harvested, it becomes part of the regenerated timber system. The regeneration system is structured differently from the system for the M01–M42 classes to permit ease in computer programming. The first nine components of δV_t are the volumes of regenerated stands by age. For example, the first component of δV_t is the volume of timber that is 35 years old at the beginning of t. The second component is the volume of regenerated timber that is 45 years old, and so on. Since the model starts with a regenerated stand 15 years of age and continues for 10 periods, it is necessary to be able to model volume for a regenerated stand 115 years of age which is the ninth component of δV_t. Once any wild mixed–conifer stand is harvested and regenerated, the deterministic model assumes that all regenerated stands are homogeneous in growth characteristics.

TABLE 1

Initial Time Stratification of Existing Timber Classes

Timber class[a]	Age	Volume	Area
	years	million cubic feet	acres
M01 (overmature)	155	6.90	600
M10 (poorly stocked)	95	51.88	11,750
M20 (mature saw)	125	193.90	26,840
M30 (multistoried)	135	450.90	45,560
M41 (young saw)	85	146.40	27,070
M42 (young saw)	225[b]	40.17	13,660
MP (plantations)	15	0.00	9,270

[a]The first six classes (M01 to M42) represent wild stands; the seventh (MP) designates recently regenerated.

[b]Even though M42 is labeled as young saw timber, it was subjectively given an age of 225 in the RAM plan so that, when coupled with its basal area estimate, the resulting volume would be close to the subjective volume estimate.

Source: Timber Resources Allocation Method.

To understand further the structure of the growth model, the role of the elements of A_t are defined. In partition form the matrix A_t is written:

$$A_t = \begin{bmatrix} & Avol_{9x9} & 0_{9x7} \\ 0_{15x17} & 0_{6x10} & Avol_{6x6} \\ & Aba_{12x12} & 0_{12x6} \\ 0_{18x15} & 0_{6x12} & Aba_{6x6} \end{bmatrix}$$

where 0_{jxk} indicates a null matrix of dimension j x k. The matrices $Avol_{9x9}$, $Avol_{6x6}$, and Aba_{6x6} are diagonal and time–varying.[1] The Aba_{12x12} is an off–diagonal matrix whose nonzero elements are $ba_{i,i-1}$ (i = 2, ..., 12). The reason the matrix is off–diagonal instead of diagonal is that, when a given element is nonzero, it has the same value for all t in which it is nonzero. Making Aba off–diagonal contributes considerably to efficiency in programming the model. The Avol matrices are multiplied by the δBA_t^2 elements of δx_t to give the volume corresponding to given basal areas which is how volumes in RAM are determined. The Aba matrices model the recursive relationship between basal area in t and basal area in t + 1. The $Avol_{6x6}$ and Aba_{6x6} model the growth of the M01–M42 condition classes, and the $Avol_{9x9}$ and Aba_{12x12} model the growth of the regenerated timber stands. The transfer of areas from the M01–M42 classes when they are harvested is modeled in the B_t matrix. Once the stands in M01–M42 are completely regeneration–harvested, $Avol_{6x6}$ and Aba_{6x6} could be discarded, and $Avol_{9x9}$ and Aba_{12x12} would represent a continuous cycling of the forest if the rotation were no longer than 115 years. For an even–aged postconversion stand, all of the elements of the A_t and B_t would be constant. This type of steady–state system is desirable from an estimation point of view since the elements of A and B could be directly and continuously updated by the data gathered by the observer. This type of estimation is discussed by Aoki (1967).[2]

The control vector, δu_t, is 20 x 1. The first six components of δu_t are for thinning on the regenerated age classes, ages 55–105. The next six components of δu_t are for thinning on M01–M42, respectively. The last 8 components are for regeneration harvests, with the 13th and 14th components being regeneration harvests in the regenerated stands and the last 6 components being for the M01–M42 classes. The model is constructed so that the b_{ij} corresponding to the *j*th components of δu_j in period t is nonzero only if the target model has a control action scheduled in that period. This restriction is made so that the LQG solution only makes adjustments in scheduled harvests; to avoid feasibility problems, it does not allow harvests at any other time. For example, let $\delta u_{t,i}$ be the

[1]The notation $Avol_{jxk}$ indicates that the particular submatrix is associated with the A matrix, and its elements are associated with volume. A similar interpretation holds for Aba_{jxk}.

[2]A comprehensive approach to parameter updating in economic LQG models remains to be thoroughly investigated. Given that the A_t, B_t, and C_t parameters are derived via a Taylor–series expansion, the relationship between A_t and the A_{t+i}, $i \geq 1$ may be very difficult to model; hence, the problem of using δz_t to estimate future parameters is most likely quite complex.

ith components of δu_t and let $u^*_{t,i} = 0$. If no restriction were made in the control model and timber stocks were below x^*_t, the $\delta u_{t,i}$ could possibly be positive, indicating $u_{t,i} < 0$, *i.e.*, the timber stocks should be increased (a negative harvest). If the assumption is made, as it is here, that actual timber stocks will tend to be less than x^*_t, then that restriction is a practical way of keeping the variables feasible. In conjunction with using output from RAM to derive the target levels, that restriction can be justified on the basis that allowing the LQG to schedule harvests for previously unscheduled areas might be inconsistent with the road or other harvest–related plans. Hence, even if a component of δx_t is nonzero, no corrective action will be taken until the RAM model indicates some cutting activity for that particular condition class. For example, RAM takes no harvesting or thinning actions on regenerated stands until they are 55 years old, so the LQG model takes no control action until a stand is 55 years old. The necessity for that restriction is a major limitation of applying an LQG model to RAM or, for that matter, to any other problem with limited feasibility regions. Additional research on more flexible ways of dealing with feasibility problems is clearly called for.

A regeneration harvest is distinguished from thinning in the model by having regeneration harvests indicate the transfer of the harvested area into the component of δx_t, $\delta BA^1_{t,1}$, which gives the level of basal area in stands that are five years old. To understand the function of B_t, it is partitioned as:

$$B_t = \begin{bmatrix} Bvol_{15x20} \\ Bba_{18x20} \end{bmatrix}$$

so that $Bvol_{15x20}$ represents the removal of timber volumes (in millions of cubic feet) and Bba_{18x20} represents removal of basal area (in thousands of square feet) as well as regeneration, a function to be described shortly. $Bvol_{15x20}$ and Bba_{18x20} are closely related since removing a volume of timber also removes basal area. For example, $u_{t,15}$ is the actual volume of timber harvested from the M01 timber class at the beginning of period t. Multiplied by the appropriate coefficient, $u_{t,15}$ is also the amount of basal area removed from $BA^2_{t,10}$, the 28th component of x_t.

An additional effect of changing the rate of regeneration harvests is an alteration in the level of regeneration. Such an alteration means that the level of basal area in the five–year–old class will change in the next period. For adjustments about the optimal values of x^*_t and u^*_t, an increase in regeneration harvests will increase the level of basal area in the five–year–old age class in the next period. The last eight components of the 16th row of B_t indicate the change in the level of basal area in the five–year basal area age class for a change in the level of regeneration harvests. The exact positioning of all elements in B_t is given in the Appendix.

Finally, e_t is a zero–mean Gaussian random variable that represents the approximate nature of the relationship in (45). It also includes the effect of omitted variables such as weather, mortality, and insect infestation.

The sampling or observing system is modeled as

$$\delta z_t = C_t \, \delta x_t + v_t \tag{46}$$

where C_t is 33 x 33, and v_t and δz_t are 33 x 1. The matrix C_t is a diagonal matrix with ones or zeroes as the diagonal elements. When a condition or age class has no possibility of existing, *e.g.*, it has been clear cut, then the corresponding diagonal elements of C_t are zero; otherwise they are one. Thus, the components of δz_t are estimates of the corresponding components of δx_t where the components of δz_t are a function solely of the sample data gathered at the beginning of period t.

In the observation system used on the Stanislaus National Forest, volume is not sampled directly, so the first 15 components of δz_t are estimates of current volumes, given the sampled estimate of basal area.[1] It can be imagined that, at the beginning of every time period, samples are taken; then estimates, based on those data, are combined with $\delta x_{t|t-1}$ using the Kalman filter to yield the optimal estimate of δx_t.[2]

Management actions in RAM and the control model are identical in timing. In RAM, control actions are taken in the middle of the decade which is the beginning of the LQG period. For example, the volume of a class is observed, the harvest is then taken, and growth for the remaining stocks in that class is then projected. To replicate this timing in the control model, the B_t matrix is written as the sum of the two matrices, $A_t\ B1_t$ and $B2_t$, and $B2_t$ is null except for the last eight elements of the 16th row which are the coefficients indicating the increase in basal area levels of the five–year–old basal area class. The matrix, $B1_t$, contains all of the basal area removal coefficients.[3] Thus, (45) can be rewritten as:

$$\begin{aligned} \delta x_{t+1} &= A_t\ (\delta x_t + B1_t\ \delta u_t) + B2_t\ \delta u_t + e_t \\ &= A_t\ \delta x_t + (A_t\ B1_t + B2_t)\ \delta u_t + e_t \end{aligned} \tag{47}$$

where the $B1_t$ and $B2_t$ elements are calculated with the removal actions occurring at the beginning of t, and A_t is also calculated at the beginning of t but after the harvest actions have been taken. In the empirical application, B_t is defined as

$$B_t = A_t\ B1_t + B2_t.$$

4.2.3. Evaluation of the LQG Model as a Representation of Harvest Scheduling

Actual implementation of the indicated harvest alterations in reference to the RAM plan is approximate. In designing the tracking model, there is a loss of information *vis–à–vis* the RAM plan. The information loss occurs because RAM disaggregates the above seven classes into subclasses. For example, the M01 class is divided into two subclasses. The growth and thinnings among the subclasses within a given class are identical. The distinction between the subclasses is that they are regeneration harvested at different times.

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[1]To derive the coefficients for A_t and B_t, volume clearly had to be measured for the regression analysis. But for the projection method laid out in (42) and (43), age and basal area indicate the actual state of the system, and volume is only an output of the system, not an input.

[2]It should be observed that A_t and B_t are not unique in their particular structure. The whole model could be cast in terms of basal area, though that would introduce considerable uncertainty unnecessarily into the objective function. The approach employed represents the planning method currently used on the Stanislaus National Forest more closely than would modeling basal area only.

[3]Since the first 15 columns of the A_t are null, there is no reason to put the –1's in $B1_{15x20}$.

The tracking model combines all subclasses within any condition class into a single class. RAM and the control model call for three types of actions: thinnings, regeneration harvests, and a combination of those two harvest activities. An information loss in using the control model occurs with the latter two activities.

When thinning is scheduled for a single timber class in RAM, every acre in that class is thinned by the same amount so that, when the LQG indicates a decrease in level of thinning in a class, it means that the thinning per acre should be decreased. Thus, there is no loss of information from simply thinning. A second situation is when only harvesting is scheduled for a class. Assume that the conditional mean, $\delta x_{t|t}$, of volume indicates that volume is 10 percent below the target level. By the assumption made in RAM that all acres within a class are homogeneous, the deviation implies that each acre has 10 percent less volume than its target level. Now suppose that the control indicates that the regeneration harvest volume from the class should be decreased by only 5 percent from the target volume. More acres will have to be harvested within the whole condition class than in the RAM plan. A loss of information occurs because the tracking model does not say from which of the several subclasses of the RAM model this additional harvest should be taken.

The third situation is when both harvests and thinnings are scheduled for a class. The LQG model is structured so that the harvest is taken, and the remaining acres are thinned. However, the LQG does not distinguish from which subclass within a particular class a change in regeneration harvest should be registered. That is, suppose that the areas to be harvested are increased, the LQG will not indicate from which subclass these acres will come.

In reference to the RAM plan, the additions or deletions of areas in the regeneration harvest plans can be taken from any of the subclasses within a class at the discretion of the manager. This is consistent with the RAM method because in the RAM model no characteristics are given to distinguish one acre from another in terms of physical or locational attributes. Thus, the loss of information *vis–ā–vis* RAM is not important from the manager's standpoint. The loss of information can be attributed, in part, to the rigidity of RAM. Thinning levels are entered as data at a prescribed level as are rotation lengths for regenerated stands. A more natural method is to let the linear programming model set the level of thinning per acre as well as to allow the program to generate rotation length as an output instead of an input. The LQG model prescribes continuous levels of removals, whereas RAM gives discrete levels so that in this respect the LQG model is less constrained than RAM.

4.3. Parameter Estimation

This section derives the values for the parameter matrices. Since the actual number of parameters estimated is large, they are not listed individually here; only the methodology for their derivation is given. The Appendix lists all of the LQG model parameters, including the target levels of x_t^* and u_t^*. The following sections derive the appropriate values for the objective function, A_t, B_t, and the required covariance matrices.

Given a complete data base, estimation of the parameters would proceed along the lines of conventional econometric methods. The objective function could be estimated directly, as in Rausser and Freebairn (1974), or by a direct specification from the U. S. Forest Service. The growth dynamics would be estimated by first specifying the appropriate structural forms and then using the reduced–form coefficients as the growth–dynamics parameters. The estimation approach is not unified, however, because of the need to construct a model using the existing data base. Should a control–systems approach be actually implemented, then data collection could be arranged to provide information necessary for using a more comprehensive estimation technique.

4.3.1. The Objective Function Coefficients

In employing the LQG technique to control a system, the measure of system performance is minimization of the disutility of not meeting the target values for all control and state variables. Given the quadratic form of the LQG objective function, the cardinal units of K_t and R_t are not important since δu_t is invariant with respect to a linear transformation of K_t and R_t. As argued in 3.3.1, the costs of uncertainty can rarely be given an exact dollar value, so the LQG analysis results in a ranking of the relative costs by the percentage of costs resulting from a given source of uncertainty.

Estimating the utility index or objective function is a problem of selecting the relative values of K_t to R_t to represent the preferences of a national forest decision–maker. An intuitive approach would be to design K_t and R_t so that they closely resemble the original preferences of the deterministic model as in the neighboring optimal control approach discussed in Athans (1972) or Bryson and Ho (1969). That approach is not used here since our interest lies partially in determining the impact of different policies. Instead, a more direct approach is taken in representing current policies and possible alternative policies since the neighboring control approach would raise considerable computational problems. In addition, it is argued shortly that the RAM objective function is an incomplete specification of management policies.

The targets used in this study were derived using RAM to maximize first–period harvests under the constraints of nondeclining, even–flow, maximum–sustained yield.[1] This policy requires that the amount harvested in any given period shall not be less than the amount harvested in any prior period. The nondeclining yield criterion is used in guiding the forest to a state of maximum–sustained yield.[2] The policy of nondeclining yield has been subjected to many criticisms of an economic nature as discussed in Zivnuska (1975). In addition to these criticisms, the policy of nondeclining yield presents considerable difficulty of implementation in an uncertain environment since the policy is not defined broadly enough to encompass uncertainty. It is clearly infeasible to guarantee any level of harvest into perpetuity so long as a forest might be decimated by fire, insects, or some other disaster. Thus, the policy must be defined more precisely with an explicit allowance for uncertainty. For example, such a policy could specify a level of probability that a given trajectory of harvests could be met into the future. The higher the level of probability or assurance wanted, the lower would be the planned harvest levels.

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[1]For a further discussion of the exact procedures to derive the harvest schedule, see U. S. Forest Service (1974b).

[2]The basic Forest Service policy has been modified since the particular RAM schedule being analyzed was derived. The National Forest Management Act of 1976 (NFMA) provides that harvest levels shall be limited ". . . to a quantity equal to or less than a quantity which can be removed from such forest annually in perpetuity on a sustained–yield basis . . ." (Public Law 94–588, Sec. 11; October 22, 1976). The NFMA also gives the Secretary of Agriculture the discretion to depart from such plans provided that such departures are consistent with the overall multiple–use objectives. More recently, the Forest Service has proposed in the National Archive of the United States (1978) that the NFMA be implemented by adhering to a nondeclining yield policy subject to various restrictions related to multiple use and economic criteria.

It is not possible to structure the LQG objective function to the exact concept used to determine x_t^* and u_t^* because nondeclining yield, as defined for deterministic models, is not a viable concept for stochastic problems. The inability to represent nondeclining yield for the LQG model is not a large hindrance because the management of national forests is also supposed to consider multiple use. Multiple use is defined as managing ". . . all of the various renewable surface resources of the National Forests so that they are utilized in the combination that will best meet the needs of the American people . . ." (U. S. Forest Service, Section 4a, 1974a). Furthermore, this combination is not necessarily the combination which yields the greatest financial return. As Chappelle, Mang, and Miley (1976) have argued, the RAM harvesting schedule tends to be biased in favor of timber activities. It seems reasonable to conclude that the current official policy for national forest management is not so precisely articulated that it can be expressed as a scalar–valued functional.

Given the numerous possible values that the objective function parameters could assume, no attempt is made to estimate one precise form or set of values for the K_t and R_t. Instead, the view of Rausser and Freebairn (1974) is adopted that constructing a unique welfare function for a given problem is both unnecessary and unrealistic. They argue that, because of the diversity of political groups and the different pressures those groups exert on political decisions, several objective functions should be constructed to reflect the extremities and compromises between the contending groups. By deriving the optimal controls for each policy, the outcomes of such policies can be examined. As Rausser and Freebairn (p. 192) suggest, "The generation of such information might even contribute to the efficiency of the bargaining process in reaching a consensus." Predicting which group will prevail or what policy function should be adopted is beyond the realm of this study or the purview of control models. The use of a set of preference functions can also be viewed as a strategy that a decision–maker can employ to measure the sensitivity of controls to variations in policies. Such a procedure defines when policy variations will have the greatest impact.

Given the above premise, the specification of values for K_t and R_t is direct. K_t can be interpreted as the loss of not having the exact stock levels for the recreational, watershed, or numerous multiple uses that publicly owned forests provide. By varying K_t relative to R_t, controls can be determined for policies that emphasize temporal existence values over harvest values and vice versa. This weighting scheme represents what Clawson (1975) perceives as the compromise that society has to make in selecting among forest utilizations that are sometimes mutually exclusive. Roughly, the relative weightings of K_t to R_t represent society's preferences for nonharvest to harvest uses. The lumping of all nonharvest multiple uses into a single loss index is a large simplification since the various multiple uses are optimized at various densities of timber stock. Thus, the losses represented by K_t must be interpreted as indices of aggregate loss. It should be noted that such a weighting scheme does not imply that higher nonharvest benefits are associated with higher timber stocks but that deviations in either direction are costly.

In the various experimental runs, the R_t is diagonal, and all of the diagonal elements of the R_t are set at one if the corresponding component of u_t^* is nonzero; otherwise the element is set at zero.[1] To obtain the effect of varied policy emphases, the K_t, $t < T$ are also diagonal. All the nonzero elements of K_t, $t < T$ within a particular policy function

[1]The R_t is then discounted for $t > 0$.

are identical except for a time discount factor. The particular value of the nonzero elements for the *i*th policy function is denoted as k_i. A diagonal element of K_t is nonzero if the corresponding element of δV_t is nonzero.[1] Given the design of the model, if a component of V_t^* is zero, then the corresponding component of V_t will also be zero irrespective of the control action; so the corresponding component of δV_t should not be weighted. One run sets k_i at zero, and then k_i is incremented in successive runs. At some point the control actions become insensitive to further increases in k_i because the effect of R_t becomes increasingly less influential; so further increases in k_i serve no informative purpose. Part of the experimental effort is directed toward determining that point. The layout of the experimental design is given in Section 5.[2]

The K_T or terminal–value matrix is specified to be diagonal, with values of unity corresponding to the volume components of V_T^* that are nonzero. These values are set arbitrarily although, as preliminary results have shown, the absolute value of the elements of K_t have very little effect on the initial period controls or the costs of uncertainty. Thus, little is to be gained by rationalizing one set of values over another. It should be observed that, with $k_i = 0$, if K_T is null, no control actions would ever be taken as can be seen in (34) and (35). Since a null K_T is nonsensical in economic terms, K_T is given a PSD value.[3] All of the K_t and R_t are discounted at 7 percent per year.

The specification that the K_t and R_t are diagonal implies an emphasis on deviations in stock or harvests within each condition class. The policy could also be one of loss from the sum of deviations in condition classes and harvest totals for the forest as a whole. In such a case the off–diagonal values of K_t and R_t would become nonzero. One would expect to balance off deficiencies in one class with relative surpluses in other classes. Results in Dixon (1976) show that there is more substitution between classes with this policy, but the other characteristics of interest in the simulations are similar to the results when the K_t and R_t are diagonal. To minimize computation costs and to allow exploration of other aspects of model design that are judged more important from a policy standpoint, the experimental policies are limited to diagonal forms for the K_t and R_t.

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[1]No loss accrues to deviations in basal area except as such deviations imply deviation in V_t since the policies are concerned with deviations in timber volumes. Thus, only the first 15 diagonal elements of the K_t may assume nonzero values.

[2]*Infra*, p. 48.

[3]When the remainder of the area in a class is to be regeneration–harvested in period t, an appropriately discounted unit value is added to the diagonal element of K_t that corresponds to the class being terminated. The value of the diagonal element of R_t corresponding to the regeneration harvest is set equal to zero. This construction reflects the fact that at $t - 1$, the policy problem is to balance the loss of deviating from current harvest levels against the combined loss in t from not having the target volume for both nonharvest and harvest uses at the beginning of t. This construction is necessitated by the fact that harvests are taken at the beginning of t, and the elements of A_t are computed after the planned harvest is taken. In a terminated class the target volumes are zero so that the corresponding coefficients of A_t and B_t are not well defined. Thus, no harvest actions are indicated for a class being terminated in its final period. The k_t value corresponding to this class must contain both the timber's harvested and nonharvested value. In the solution algorithm the level of the class in its terminating period is regarded as the final harvest level.

4.3.2. The Growth–Dynamics Parameters

This section considers estimation of the A_t and B_t parameters. Recall that in Section 4.2.1 the original RAM growth system could be characterized by two equations instead of three. In U. S. Forest Service (1974b), height is specified to be a linear function of the base 10 logarithm of stand age, AG_t, in decades. Volume is then given in cubic feet per acre as:

$$Vol_t = \alpha + \beta\ (HT_t \cdot BA_t)$$

where HT is height in feet and basal area and BA_t is basal area in square feet per acre.[1] To eliminate the height equation, the volume equation is respecified as:

$$Vol_t = \alpha + \beta\ [(\log_{10} AG_t) \cdot (BA_t)]. \tag{48}$$

When regressed on the original data, (48) is

$$\underset{}{Vol_t} = \underset{(420.7)}{-685.86} + \underset{(2.01)}{36.36}\ [BA * \log_{10} (AG_t)] \qquad R^2 = .90 \tag{49}$$

with the standard error of the estimates in parentheses. The R^2 of the original volume equation in U. S. Forest Service (1974b) is .91 so that little is lost in terms of explanatory power by respecifying the volume relationship. Using the F test suggested by Goldfeld and Quandt (1965), (49) is heteroscedastic at a 95 percent confidence level. Reestimation after dividing the dependent and independent variables in (49) by the square root of the independent variable gives the result

$$Vol_t = \underset{(104.77)}{-119.17^*} + \underset{(1.322)}{33.183^*}\ [\log_{10} (AG_t) \cdot BA_t] \qquad R^2 = .89,\ \ s.e. = 97.48 \tag{50}$$

where the asterisk denotes that the coefficients are estimated by generalized least squares. This implies that the variance of the dependent variable is proportional to the independent variable. In (50) the R^2 is slightly lower which is to be expected with heteroscedastic data since the majority of the variance in the dependent variable can be generated by a minority of the observations.

The U. S. Forest Service (1974b) plan gives none of the variances of the coefficients or standard errors. Since the LQG model needs those statistics when coefficient uncertainty is recognized, it was necessary to reestimate the basal area growth equation. Using the same functional form and data as in U. S. Forest Service (1974b), the regression equation is:

$$\begin{aligned} BAG_t = & \underset{(11.6)}{-20} - \underset{(.0935)}{.2034}\ BA_t + \underset{(2.117)}{3.963}\ (BA_t)^{1/2} + \underset{(65.22)}{75.55}/AG_t \\ & + \underset{(.6194)}{2.028}\ BA_t/AG_t - \underset{(12.83)}{15.956}\ (BA_t)^{1/2}/AG_t \qquad R^2 = .88,\ \ s.e. = 4.119 \end{aligned} \tag{51}$$

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[1] As in U. S. Forest Service (1974b), all of the stands are assumed to be Dunning Site II.

where BAG_t is the basal area growth per acre in square feet. The estimates differ slightly from those in U. S. Forest Service (1974b). The determinant of the correlation matrix of the independent and dependent variables is 1.9×10^{-6}, suggesting a degree of multicollinearity such that rounding error could account for the differences between the results.

Given the greater efficiency of the estimates in (50) and the necessity for additional regression results not reported in U. S. Forest Service (1974b), the parameters in (50) and (51) are used in lieu of those used in the original RAM run. Since the volume regression results differ substantially from those used in the RAM run to derive the x_t^* and u_t^*, the targets have to be updated.[1] The RAM model could, of course, be run with new data. Alternatively, the current RAM plan could be employed by following the same harvest schedule in terms of acres harvested and thinned but computing the volume per acre in light of (50) and (51). This latter option is taken because the former method entails substantial practical difficulties. For the purposes of this study, the latter method is quite satisfactory since the qualitative results of the study will not differ regardless of which method is pursued.

Somewhat parenthetically, it may be observed that many variables such as mortality or weather are not included in (50) and (51) due to a lack of sufficient data. A stochastic model makes a much better accommodation for such misspecifications than deterministic models. By using an observer at the beginning of each period, the changes in the timber stocks due to forces not explicitly in the growth dynamics are picked up by the observer. Even though some significant variables are formally excluded, their actual impact is thus consistently incorporated into the LQG model. Thus, the problem of incomplete specification is mitigated.

The basal area transition coefficients, Aba, are derived directly from (51). Adding BA_t to both sides of (51) gives the needed recursive relationship:

$$BA_{t+1} = BAG_t + BA_t. \tag{52}$$

To determine the approximate linear coefficients, the first derivative of (52) is taken with respect to BA_t so that the nonzero elements of the Aba are defined as

$$\begin{aligned} Aba_{ij} = &-.2034 + 1.982\ BA_t^{-1/2} + 2.028/AG_t \\ &- 7.979\ (BA_t)^{-1/2}/AG_t + 1 \end{aligned} \tag{53}$$

where both BA_t and AG_t are defined at their target values.[2] Recall that A_t is determined after the scheduled harvests have been deducted from x_t^*.

The state–transition parameters for the volume states, $Avol_t$, are derived from (50). Relation (50) may be written as

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[1]The volume regression reported in U. S. Forest Service (1974b) is $Vol_t = -476 + .26\ (HT_t \cdot BA_t)$.

[2]RAM does not give target values for basal area per acre for ages 5, 15, and 25. Respective subjective estimates for these age groups based on information supplied by Klaus Barber, at that time forester on the Stanislaus National Forest, are 5, 30, and 65 square feet of basal area.

$$\text{Vol}_{t+1} = -119.17 + 33.183\ [F\ (BA_t) \cdot \log_{10}\ (AG_{t+1})] \tag{54}$$

but by the definition of BA_{t+1} in (52), $BA_{t+1} = F\ (BA_t)$ so that

$$\text{Vol}_{t+1} = -119.17 + 33.183\ [BA_{t+1} \cdot \log_{10}\ (AG_{t+1})] \tag{55}$$

which gives the recursive relationship desired. By taking the first derivative of (55) with respect to BA_t, the coefficients of Avol are derived. Since 33.183 and AG_t are constant with respect to BA_t, the typical nonzero element of Avol is

$$\text{Avol}_{ij} = .033183 \cdot \log_{10}\ (AG_{t+1}) \cdot \text{Aba}_{i+18,\,j})$$

where the coefficients have been adjusted to give volume in millions of cubic feet related to basal area in thousands of square feet. Table 2 gives the coefficients for A_0.[1]

TABLE 2

State Transition Coefficients for A_0 for the Initial Period[a]

Nonzero coefficients	
$a_{10,28}$ = .0415 (.18x10^{-2})[b]	$a_{18,17}$ = 1.5392 (.299)
$a_{11,29}$ = .0375 (.15x10^{-2})	$a_{28,28}$ = 1.0247 (.15x10^{-1})
$a_{12,30}$ = .0395 (.16x10^{-2})	$a_{29,29}$ = 1.1065 (.71x10^{-2})
$a_{13,31}$ = .0401 (.17x10^{-2})	$a_{30,30}$ = 1.0534 (.12x10^{-1})
$a_{14,32}$ = .0361 (.15x10^{-2})	$a_{31,31}$ = 1.0416 (.13x10^{-1})
$a_{15,33}$ = .0493 (.24x10^{-2})	$a_{32,32}$ = 1.1131 (.93x10^{-2})
	$a_{33,33}$ = 1.0838 (.31x10^{-1})

[a]The element of the *i*th row and *j*th column of A_0 is indicated by $a_{i,j}$.

[b]Numbers in parentheses indicate standard errors.

Source: Computed.

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[1]When AG_t = 5, (53) gives a value of −2.40. Subjectively, that value is set at 1.65 to be more reasonable with respect to the values of (53) in later years.

When acknowledging the uncertainty of the parameters in the LQG solution, the variances of the a_{ij} must be estimated. Those variances can be computed by observing that the linear terms of the Taylor–series expansion of (51) are the regression coefficients multiplied by a vector of independent variables. In deriving the Aba, AG_t and BA_t are assumed to be known so that the relevant linear regression coefficients of (51) are multiplied by the target values of AG_t and BA_t and then summed to get the linear coefficient as in (53). The covariance matrix of the coefficients is available from the regression results. Since the derived coefficients for the LQG model are linear combinations of the regression coefficients, the variances of the elements of A_t are derived by pre– and postmultiplying the relevant rows of the covariance matrix of (47) by the vectors that give the elements of Aba_{12x12} and Aba_{6x6}. The variance for the linearized coefficients in (55) is derived by observing that the coefficient is the product of the regression coefficient (0.033183), $\log_{10}$ (AG_{t+1}), and the linearized coefficient for BA_t. Because 0.033183 and the linearized coefficient of BA_t are random variables, the product rule for moments of random variables is used to obtain the variance for the elements of Avol.[1]

The basal area removal coefficients, Bl_t, cannot be obtained directly from (51) and (54) because no harvesting activity is included in those relations. Thus, Taylor's theorem cannot be applied directly. A practical approach is to use (51) and (54) to determine the amount of basal area being removed for any control action. By using the optimal values of BA_t and the given AG_t in (51) and (54) and assuming that the removal actions are linear as they affect their states, the resulting coefficients are similar to those that would be obtained by a Taylor series if relations (51) and (54) included removal actions.

Recall that, to obtain B_t, the state–transition matrix, A_t, is multiplied by $B1_t$, and the product is added to $B2_t$. Since the first 15 columns of A_t are null, only the coefficients relating to removal of basal area and regeneration need be estimated for B_t. The elements of $B1_t$, relating to the removal of basal area, are determined by computing how many thousands of square feet of basal area correspond to the removal of 1 million cubic feet of timber from a given timber class defined as the optimal basal area. For example, in the initial period the thinning of 1 million cubic feet of timber from the M01 timber class requires a corresponding removal of 25.317 thousand square feet of basal area. Algebraically, letting $b1_{ij}$ be the *ij*th element of B1, they are defined as

$$\frac{1}{\text{Vol*/acre}} \cdot BA_t^* = b1_{ij}$$

where Vol*/acre is volume in millions of cubic feet and BA* is in thousands of square feet per acre. Basal area and volume are starred to indicate that they are the target values derived from the RAM targets. For the coefficients denoting thinning, the constant term in (50)—–used to calculate Vol*/acre—–is ignored in order to be consistent with the method used to determine basal area removal in the RAM model.

The $B2_t$ coefficients are those that apply to the change in basal area in the five–year–old class, *i.e.*, those coefficients that imply a change in the regeneration schedule due to a change in the harvest schedule. These are derived by determining the number of acres that corresponds to removal of an additional million cubic feet in any period

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[1] The volume and basal area relations are estimated by single–equation methods, so the two random coefficients are assumed to be independent.

given BA_t^*. The number of acres is then multiplied by 0.005, the basal area in thousands of square feet that corresponds to the change in basal area in the five–year–old age class. Algebraically, letting $bl_{16,j}$ be the *j*th element of the 16th row of Bl_t,

$$\frac{1}{\mathrm{Vol}^*/\mathrm{acre}}\ 0.005 = bl_{16,j} \qquad j = 13, \ldots, 20.$$

The variances of the elements of the Bl_t and $B2_t$ are determined in an approximate way. Recall that the elements of Bl_t are determined by deriving a target level of basal area per acre to be removed, BA_t^*. Using BA_t^*, Vol/acre is predicted to derive the elements of Bl_t. Thus, Vol/acre can be viewed as a random variable, with its variance being the forecast variance.[1] Now, BA_t^*/(Vol/acre) can be viewed as the quotient of two random variables in which the BA_t^* is a degenerate random variable. As shown by Mood, Graybill, and Boes (1974), there is no analytical expression for the moments of quotients of random variables; but the linear term of a Taylor–series expansion gives the mean as $E(x/y) = \bar{x}/\bar{y}$, and the Taylor–series expansion for the variance, truncating after the second–order terms, is

$$\mathrm{Var}\ (x/y) = \left(\frac{\bar{x}}{\bar{y}}\right)^2 \frac{\mathrm{Var}\ (x)}{\bar{x}^2} + \frac{\mathrm{Var}\ (y)}{\bar{y}^2} - \frac{2\mathrm{cov}\ (xy)}{\overline{yx}}.$$

These approximations are used to determine the moments of the elements of Bl_t and $B2_t$.

Finally, to obtain B_t, A_t is multiplied by Bl_t and the product is added to $B2_t$. The variances of the resulting B_t coefficients are derived by assuming that the elements of A_t and B_t are independent and then using the product rule. A further simplification is to assume that every element of A_t is distributed independently of every other element in A_t and B_t. A similar assumption is made for the elements of B_t.

In total, there are 186 nonzero A_t parameters and 132 nonzero B_t parameters. All but nine of these A_t parameters are at least four times their standard error. The remaining nine parameters are less than twice their estimated standard error. The nonzero B_t parameters are statistically less significant. Of the 132 parameters, only 41 are more than four times their standard error, while 19 are between twice and four times their standard error. The remaining 72 parameters are less than twice their standard error. Nonetheless, 65.4 percent of the total parameters in A_t and B_t are at least four times their standard error; only 25.5 percent are less than twice their standard error. By extending Prescott's (1972) findings, one can conjecture that A_t and B_t can be assumed constant and still derive an approximately optimal solution to the LQG problem. This hypothesis is empirically examined in Section 5.[2]

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[1]The forecast error for the linear model $y = X\beta$ for some y^* given X_* is $s^2 + s^2 X_* (X'X)^{-1} X'_*$, where s is the standard error of the regression.

[2]*Infra*, p. 48.

4.3.3. Covariances of the Error Terms and Initial Conditions

The LQG method requires three sets of covariances: $P_{0|0}$, the initial covariance of the states; Ω_t, the covariance of the equations of motion; and θ_t, the degree of observer precision. Since the equations in the growth dynamics are estimated by single–equation methods, the covariance matrices are nearly diagonal. For Ω_t, it is specified that the error term of a volume equation is distributed jointly with the error term of the basal area growth equation for the respective class being modeled. Similarly, since the observation of volume is conditional on basal area, it is assumed that, within any one class, basal area and volume observations are correlated. The same assumption holds for $P_{0|0}$ since it is regarded as the observer precisions at the beginning of the initial period and calculated in the same way as the θ_t. However, the observation errors within any given class are assumed to be independent of errors in any other class. It is also assumed that Ω_t and θ_t are white–noise processes, *i.e.*, serially independent. The white–noise assumption implies that the sampling units are selected randomly at each point in time. Furthermore, it is assumed that the basal area within each sampling unit is measured perfectly.

The error terms of the first 15 state equations in each period are specified as the sum of two uncorrelated random variables: the error of the function giving volume as a function of BA_t and AG_t and the error of the function projecting BA in t + 1 given current basal area and age. This latter error must be incorporated into the first 15 components of e_t because current volume is a function of current basal area. The variances of these errors are obtained from the linearizations of (50) and (51). The determination of the variances of the linearized versions of (50) and (51) is difficult since both are nonlinear and neither contain any control actions (harvest activities). Even so, given that age is considered a known parameter in the model, (50) is linear in basal area so that the standard error of the regression in (50) is a very reasonable estimate for the standard error of the linearization of (50).

For the linearization of the state equations representing basal area growth, a similar approach is taken. First, the independent variable, $BA_t/A1_t$, in (51) has a correlation coefficient of 0.922 with BAG_t, and the correlation coefficient of all of the variables is 0.936. It can be concluded that most of the variation in BAG_t can be explained by $BA_t/A1_t$. Thus, the standard error of (51) for the linearization of (51) is a good approximation to the actual unknown standard deviation. The standard error of (51) is used as the basis for computing the last 15 diagonal elements of Ω_t.[1] BA_{t+1} is also in the linearized volume equation, multiplied by $.033183 \cdot \log_{10} AG_{t+1}$, so the additive error associated with BA_{t+1} becomes part of e_t multiplied first by $33.183 \cdot \log_{10} AG_{t+1}$. Thus, e_t for the first 15 equations is the sum of two uncorrelated random variables: the error due to estimating volume given basal area and the error in estimating basal area in t + 1, given basal area in t multiplied by a constant. The variance of e_t is then computed using the rule for the variance of the sum of two random variables.

Since basal area enters the volume equation additively and is multiplied by a constant, the covariance between the respective volume and basal area classes is the variance of

[1]For basal area in its first three periods of growth, the standard error of (51), 4.119, gave unreasonably large variances. Thus, the standard errors were subjectively set at 1, 2, and 3 for the 5 –, 15–, and 25–year–old basal area growth equations, respectively.

the basal area equation multiplied by .033183 · $\log_{10} AG_{t+1}$. For example, let *a* and *b* be two uncorrelated random variables with variances σ_a^2 and σ_b^2 and k is some constant. Then

$$\text{Cov } [(a + kb), b] = k\sigma_b^2 \ .$$

Because all the acres within a class are considered to be homogeneous, the variances per acre are multiplied by the square of the number of acres in that class and normalized into the units of the components of δBA_t^1 and δBA_t^2. This is done for all $P_{0|0}$, Ω_t, and θ_t; and it results for all of the covariance matrices being heteroscedastic.

The covariance matrices of the observer relationship for basal area must be specified by the decision–maker since no observer precision levels are given in RAM. Most volume estimates on a per acre basis in U. S. Forest Service (1974b) have a standard error between 5 and 11 percent of the sample mean. It is postulated that the future observations on basal area will be such that the standard error of the estimate will equal 5 percent of the sample mean. Thus, given the target levels of basal area, it is possible to estimate the last 18 diagonal elements of θ_t. For example, at the beginning of the first period in the M01 timber class, the RAM plan indicates 100 acres with a total basal area of 24.212 thousand square feet. Its standard error to the required observation precision is 1.21 so that its observation variance is 1.48.

Since the estimate of current timber levels are conditional upon basal area estimates, it is specified that the observation variance is the sum of the regression variance in (50) plus the variance of basal area observation times the square of the product of the slope coefficient and $\log_{10} (AG_t)$, *i.e.*, the typical nonzero diagonal element of the first 15 rows of θ_t, denoted θ_{ii}, is

$$\theta_{ii} = s^2 + [33.183 \cdot \log_{10} (AG_t)]^2 \ \theta_{i+18,i+18} \tag{56}$$

where $\theta_{i+18,i+18}$ is the observer variance of basal area and s is the standard error of the regression (50). Of course, (56) is adjusted for units and heteroscedasticity. The covariance of the two error terms is simply the covariance of the sum of two uncorrelated random variables and one of the original variables where this latter random variable has been multiplied by a constant in the sum. The formula is the same as for the covariances in Ω_t.

The initial covariance of the basal area states is difficult to determine. The initial state estimate was determined by dividing the original classes into various strata and sampling each stratum for basal area.[1] In each stratum apparently only one observation was taken. The mean for the whole class was calculated by combining subjective estimates with a weighted mean of the observations. Since no further information is available, the assumption is made that the standard error of the initial estimate of basal area is equal to 5 percent of the estimated mean for a given age class so that $P_{0|0}$ is computed in the same way as the θ_i.

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[1]These are actually data generated by the sampling for volume done for the whole forest on a probability proportional to size (volume) basis.

4.3.4. Error Introduced by Linearization

In this section the question is addressed as to how well the linearized dynamics system (41) duplicates the nonlinear dynamics (46) and (47). It is not necessary to validate the observer system since it is essentially identical in design to the system in U. S. Forest Service (1974b). It is important to determine how closely the linearized growth system approximates the nonlinear system so that the size of the neighborhood in which the LQG is accurate can be determined. Clearly, when all of the actual levels are met, the linearized system exactly duplicates the nonlinear system. As the actual path of the system diverges from the targets, the accuracy of the linearized model will deteriorate.

The method used for validation was to vary the actual harvest levels from 2.5 percent above and below the target harvest to 50 percent above and below. The variation of a particular harvest between the given limits was given by a uniformly distributed random variable. The error terms in both the linearized and nonlinear regressions were set equal to zero, since random events would affect both systems almost identically, given that both of the specifications posit that the random disturbance is additive.

Six levels of possible control variation were set between the limits cited above, and 12 runs were made—two for each level of variation. Bivariate regressions were obtained for each run letting the state of the system given by the nonlinear dynamics be a function of the state levels predicted by the linearized model. In a perfect replication the constant terms would be zero, the slope coefficients unity, and coefficients of determination equal to one.[1]

The results demonstrate that the linearized model is a very good model of the nonlinear relationship. The coefficients of determination range from 1.0 to .81. Three of the slope coefficients were different from one at the 95 percent confidence, and three of the intercept terms were statistically different from zero at the 95 percent confidence. These results are not surprising since volume as a linear function of basal area given age and the growth of basal area over time is relatively smooth so that linearizing it over decades gives a reliable approximation.

The accuracy of the linearized model is an important result in itself. The Kalman filter could be used for timber stock estimation regardless of whether an optimal control model was being used to set harvest levels. The limiting concern when the filter is used alone is the accuracy of the linearized model of the growth dynamics. Since the linearized model in this case appears to be quite accurate, the Kalman filter can be recommended for use in stock estimation with the Stanislaus National Forest growth model. Additionally, errors made by a linearized model are partially corrected by the observer and the update phase of the Kalman filter. Given these results, it appears that linear recursive estimators have much to offer to forest mensuration problems.

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[1]This simulation is not a stochastic simulation in the sense of regressing actual observed values on values stochastically predicted by the model as discussed in Aigner (1972). In the bivariate regressions above, the additive error term represents solely the error introduced by linearization, and it is assumed in concordance with the LQG theory that such error is normally distributed. Due to the lack of time series data, it was not possible to compare the LQG model with observed data.

4.3.5. Characteristics of the Linearized System

Engineering control literature describes a number of the characteristics of linear models that are used to assess the qualitative nature of the models. Among these properties are stability, controllability, and observability. This section analyzes the estimated parameters for A_t, B_t, and C_t with respect to the above characteristics for their relevance to the LQG model just estimated.

A very important property of any system is the stability of the state–transition matrices, A_t. If a system is unstable—given some initial $\delta x_0 \neq 0$—δx_t, as t approaches infinity, need not converge to the origin. It can be immediately observed that the system is not stable since, for all t, the 32,32 element of all the A_t is greater than one, and there are no other nonzero elements in row 32 of the A_t. Thus, left unperturbed, a given deviation will continue to grow.

To combat any explosive tendencies in the system, control effort can be exerted. A system is said to be controllable if it is possible to move it to any predetermined state in a finite number of periods. The estimated system is not controllable over the 10–period trajectory. This lack is a direct result of the construction of the A_t and B_t. They are structured so that in all periods the Euclidean space spanned by the A_t and B_t is strictly less than 33. That is to say, by design of A_t and B_t it is impossible to move the system from any given initial state to any desired state within the 10–period time horizon.

The lack of controllability is due, in part, to the restrictions on the elements of the B_t. Recall that, to avoid feasibility problems on the variables, the tracking model only adjusts harvests indicated by RAM; it does not initiate any. If that restriction could be lifted, the manager could exert much more control. However, even with all possible controls, the forest can only be moved to levels that are biologically feasible.

The last property of linear systems to be discussed here is observability. This property implies that, given enough observations, δz_t, it is possible to determine the initial state, δx_0. This property is not relevant for the empirical model because it is assumed that an estimate of the initial state is known; hence, the property of observability is meaningless in this particular application.

All three of the above properties have been defined as deterministic concepts. When the true stochastic properties of the growth dynamics system are recognized, all of the above properties are defined in ways that are quite different. In general, stochastic observability, stability, and controllability[1] analyze the boundedness of $P_{t|t}$ over time. These advanced topics in control theory, covered in Aoki (1967), are difficult to analyze with time–varying covariance matrices, Ω_t and θ_t. Empirically, the results show that $P_{t|t}$ does not seem to exhibit explosive tendencies.

4.4. Summary

In Section 4 it was shown that the usefulness of the LQG technique for analyzing and studying stochastic problems in harvest scheduling depends directly on how accurately

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[1]For further information on observability, stability, and controllability, see Meditch (1969).

the deterministic model of the underlying optimization problem can be translated into the required linear–quadratic forms. Three major difficulties are encountered in fitting the Stanislaus National Forest RAM plan into LQG form. The first is that growth and timber removal actions are entered as tabular data as opposed to continuous functions; the second difficulty is that the objectives of national forest management are not clearly defined for a stochastic problem; and the last difficulty is that a timber–harvesting problem requires extensive use of feasibility constraints.

The difficulty inherent in the tabular form of the RAM input is handled by using the original functions employed in obtaining the tabular data and employing Taylor's theorem. The resulting model of growth dynamics offers both advantages and disadvantages over the RAM model. A desirable aspect of the LQG dynamics is that thinning levels and harvest rotation become, in most instances, outputs of the solution instead of inputs to it. A drawback to the LQG dynamics is that there is a loss of information *vis–à–vis* RAM.

The problem of feasibility constraints is partially solved by not allowing variables with a target value of zero to be altered in the solution. Given that small variations in timber removals at a zero level might seriously disrupt long–range plans and that the variations would be small relative to the harvest levels, this approximation is judged acceptable. Since it does not assure feasibility in all cases, however, the behavior of this aspect of the model remains to be examined in Section 5 below.

The LQG objective function is hindered by a precise articulation of Forest Service objectives under uncertainty. The policy of nondeclining yield is not well defined for an uncertain environment, and the multiple–use criterion is vague. Given these circumstances, a set of objective functions is postulated to span the reasonable possibilities of diagonal quadratic preferences. In the context of forestry, this results in various weightings of harvest versus nonharvest uses if the decision–maker is risk averse.

A final note concerns use of the Kalman filter. Its implementation is not tainted by feasibility or objective function estimation problems. The only part of the filter significantly affected by the linearizations is the growth dynamics. The experimental results indicate that the linearizations are an accurate model of the underlying nonlinear model. Thus, the Kalman filter can be used as an estimator for the Stanislaus National Forest timber stocks even if the rest of the LQG control technique is judged inappropriate.

5. ANALYSIS OF SIMULATED RESULTS

This section presents and discusses the results of several simulations with various stochastic assumptions and policy directions. The objectives are dual in nature: (1) to demonstrate how stochastic control models can be used to answer problems that confront the decision–maker and (2) to show how optimal actions change as objectives of management change. This latter point measures the sensitivity of control actions to policy variations. The more sensitive control actions are to policy variations, the more important it is to lessen uncertainty about society's preferences for national forest management. This section is divided into three parts. Section 5.1 examines the control actions and resulting state trajectories under varying policies and stochastic specifications, Section 5.2 demonstrates the use and significance of the Kalman filter by analyzing the reduction in variance of the estimates of timber stocks compared with the use of nonsequential estimators, and Section 5.3 analyzes the costs of the different sources of uncertainty and determines the impact of more precise information.

By varying the objective function parameters, it is possible to determine whether changes in K_t and R_t affect some aspects of managerial actions more than others and whether the range of relative values between K_t and R_t can be determined to establish when harvest objectives take precedence over having desired stocks and vice versa in the LQG model. An important question to be answered in this section is whether it is a serious misspecification to assume A_t and B_t to be constant when they are actually random. That comparison is important since it indicates when a certainty–equivalent assumption (setting random parameters equal to their means and optimizing) is acceptable.

Additionally, from (34), (35), and (37) it can be seen that the costs of uncertainty vary with the value of K_t and R_t. By varying the K_t values relative to R_t, it is possible to calculate when a given source of uncertainty is more costly and how various sources of uncertainty vary in cost, given the same objective function. A final point of investigation is the impact of lessening uncertainty. From evaluation of the expected cost of uncertainty in the objective function (37), it is clear that the size of θ_t and Ω_t—the observer variance and growth variance—through their effect on $P_{t|t}$ and relation to H_t are important in determining the costs of uncertainty. Varying θ_t or Ω_t allows evaluation of the impact of greater precision on the costs of uncertainty.

In addition to the above areas, attention is devoted to assessing the use of the LQG model with respect to variable feasibility constraints. This is very important since variable constraints play an active role in micromodels, and one of the objectives of this study is to determine whether the LQG can be used effectively with micromodels.

Simulation of the real time solution of the LQG model requires generating a series of volume and basal area observations. The δz_t are obtained by generating a series of e_t and v_t and initial error on $\delta x_{0|0}$. Given the construction of $P_{0|0}$, θ_t, and Ω_t, the model implies a bivariate normal distribution between a class volume and basal area. Since the error term of the first 15 equations of δx_t is the sum of .033183 $\cdot \log_{10}(AG_t)$ times the error of basal area growth and the error term in the volume equation (50), the first 15 e_t's are calculated to reflect this. Simply, random normal deviates are generated for δBA and the error for the error term in (50). The former error is multiplied by .033183 $\cdot \log_{10}(AG_t)$ and added to the latter deviate. This makes the calculation of the error term consistent with the construction of θ_t and Ω_t as discussed in 4.3.3. Thus, δx_t, the "true" state is computed as:

$$\delta x_{t+1} = A_t \, \delta x_t + B_t \, \delta u_t + e_t$$

and δz_t is computed as:

$$\delta z_t = C_t \, \delta x_t + v_t .$$

There are 16 different simulations labeled S1–S16. The exact design of these simulations is in Table 3. This section summarizes the results. A more detailed listing of the results is given in a separate supplement of results.[1]

To facilitate rough comparisons between simulations, the same random variates are used in S1–S10. In S11 through S16 the variates are calculated according to the covariance specification of the particular simulation. However, control responses are strictly comparable between any two simulations only for the initial period. The simulations are differentiated by varying the coefficients in the objective function, the stochastic assumptions on A_t and B_t, and the levels of Ω_t and θ_t.

For the total problem, run in single precision, processor time was 4.9 seconds.[2] About two–thirds of this time is devoted to obtaining the filter covariance matrices. If none of the elements in A_t, C_t, θ_t, Ω_t, or $P_{0|0}$ are changed, then the filter need be computed only once and stored on disk for successive runs. Alternatively, if sensitivity analysis is focused on θ_t and Ω_t, then only the filter problem needs to be rerun, with the H_t, A_t' H_{t+1} B_t, G_t, and G_t being stored on disk.[3]

5.1. Examining the Harvest and Volume Trajectories

The real time operation of the model is illustrated by examining in detail the initial–period results for two simulations with differing objective functions in 5.1.1. The results of S1–S10 are aggregated into summary measures in 5.1.2 to analyze the performance of the LQG over the 10–period horizon from both modeling and policy points of view. The effect of random A_t and B_t on the control and state trajectories is discussed in 5.1.3.

5.1.1. Initial–Period Response for S1 and S4

In all of the simulations, it is assumed that M01, M30, and M42 timber classes have the following deviations from their target volumes, defined by RAM at t = 0.

Timber class[4]	Target volume	Estimated	$\delta x_{t\|t}$
M01	6.90	6.89	.01
M30	450.90	448.09	2.81
M42	40.17	32.32	7.85

[1]The supplement with the full set of results is obtainable from the authors on request.

[2]The simulations were run on a CDC 7600 at the University of Illinois at Urbana–Champaign.

[3]The program, written in FORTRAN IV, uses matrix subroutines developed by White and Lee (1971).

[4]For definitions of timber classes, see Table 1, *supra*, p. 31.

TABLE 3

Characteristics of the Experimental Runs[a]

							θ_t			Ω_t	
Simulation	$K_t = 0$	$K_t = .10$	$K_t = .20$	$K_t = 1.0$	$K_t = 4.0$	A_t, B_t assumed known	At initial values	Reduced by 75 percent	Last 18 diagonal elements reduced by 75 percent[b]	At estimated values	Last 18 diagonal elements reduced by 75 percent[b]
S1	x					x	x			x	
S2		x				x	x			x	
S3			x			x	x			x	
S4				x		x	x			x	
S5					x	x	x			x	
S6	x						x			x	
S7		x					x			x	
S8			x				x			x	
S9				x			x			x	
S10					x		x			x	
S11	x					x		x		x	
S12				x		x		x		x	
S13	x					x			x	x	
S14				x		x			x	x	
S15	x					x			x		x
S16				x		x			x		x

[a]The x indicates that a run has the indicated characteristics.

[b]Since the remaining nonzero elements in these covariance matrices are partially a function of the last 18 diagonal elements, the remaining nonzero elements are accordingly adjusted using the procedures in 4.3.3, *supra*, p. 44.

Source: Computed.

The values in the "estimated" column are assumed to be the conditional mean estimates of the existing volumes.[1] The adjustments to the thinnings and regeneration harvests for S1 are mild as shown in Table 4. The preferences in S1 strongly emphasize maintaining harvest flows. All of the harvests and thinnings in volume classes that are below their target levels remain at their target levels or decline as would be expected. A harvest reduction is taken in M42 because of the large relative discrepancy between target level and actual level in M42. In contrast to S1 where the K_t, $t < 10$, are set at zero, in S4 where $K_t = 1$, the control responses are comparatively more vigorous. In the M42 class for S4, the ratio of the deviation in harvest to the deviation in volume is 0.60, whereas in S1 it is 0.04. The increased responsiveness is due to the importance of having current stock levels as implied by the objective function in S4.

In S1 the potential decline of harvests is postponed to a less costly future. However, when a loss is associated with not having desired current stocks, say for multiple–use purposes, then no opportunity exists for postponing losses to a less costly future. The conclusion cannot be drawn that assigning temporal existence values to volume deviations will always result in decreasing harvests. Recall that the control rule, $\delta u_t = -G_t \ \delta x_{t|t}$, is linear in $\delta x_{t|t}$. In the particular example displayed in Table 4, all of the volume deviations are below their targets. If the sign of $\delta x_{t|t}$ were changed, then the sign of δu_t would also change, indicating that for S1 and S4 most of the harvest activities would be above their target levels in the initial period.

The problems presented by binding variable constraints can be seen in Table 4. Given the restrictions imposed on the elements of B_0 in the LQG model, no control or state variables lie outside their feasibility region for the initial period of S1 or S4. If this restriction were not made, the number of feasibility constraint violations could be considerable. For example, assume that no regeneration harvest activity is scheduled for the M42 class in the initial period of S1 but that the possibility for regeneration harvest in M42 is permitted in B_0. The corresponding element of δu_0 would still be 0.353 indicating a negative harvest. In a forestry context this could be interpreted as an indication to increase the timber stock as through fertilization or planting. The restriction on the B_t is not totally effective in preventing feasibility violations. For example, assume that $\delta x_{0|0}$ is multiplied by 5.12 so that the 15th component of $\delta x_{0|0}$ is 40.19, essentially indicating that there is no timber in the M42 class. In S1 the control would indicate a harvest of 11.86 which is clearly impossible since no timber exists. Feasibility violations are rare in the simulated results. In S1–S16 there are no more than three control violations per simulation; this is out of a possible 60 control actions in each simulation. Violations tend to occur when the level of the target harvest is close to the level of the target volume of the given class. Hence, in periods where the remainder of an age or timber class is to be completely clear–cut, feasibility violations are frequent. In the simulations the infeasible harvests are set equal to their nearest feasible level.

The conclusion that can be drawn from the above results is that feasibility violations are not a serious problem when variables remain relatively close to their target values. This implies that the LQG method can be used even if the original optimization problem

[1]In general, it is to be expected that the initial state will be below the target level since a system is rarely "on target" at the beginning. Additionally, with $x_t^* \neq x_0$, the resulting initial period controls are the solution of a more realistic stochastic control problem instead of the deterministic controls, u_0^*.

TABLE 4

Initial Period Harvests for Simulations One (S1), Four (S4), Six (S6), and Nine (S9)

Timber class[a]	S1 Scheduled Regeneration harvest	S1 Scheduled Thinning	S1 Actual Regeneration harvest	S1 Actual Thinning	S4 Scheduled Regeneration harvest	S4 Scheduled Thinning	S4 Actual Regeneration harvest	S4 Actual Thinning
	million cubic feet							
M01	5.75	0.31	5.75	0.31	5.75	0.31	5.76	0.31
M10	13.28	0.00	13.28	0.00	13.28	0.00	13.29	0.00
M20	20.95	0.00	20.95	0.00	20.95	0.00	20.95	0.00
M30	10.89	86.77	10.89	86.77	10.89	86.77	10.04	85.93
M41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M42	13.66	0.00	13.31	0.00	13.66	0.00	8.99	0.00
	S6				S9			
M01	5.75	0.31	5.76	0.31	5.75	0.31	5.78	0.30
M10	13.28	0.00	13.28	0.00	13.28	0.00	13.32	0.00
M20	20.95	0.00	20.95	0.00	20.95	0.00	20.98	0.00
M30	10.89	86.77	10.89	86.77	10.89	86.77	10.05	85.96
M41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
M42	13.66	0.00	13.31	0.00	13.66	0.00	9.32	0.00

[a]The classes (M01 to M42) represent wild stands.

Source: Computed.

has variables with constrained feasibility regions. Penalty functions would also prevent feasibility violations but would make interpretation of the costs of uncertainty difficult if not meaningless. Also, computational costs would be increased substantially. Hence, for economic problems, the use of restrictions on the B_t may be a better and more useful approximation.

The initial–period control responses show little substitution effect among the controls.[1] This is a direct result of the diagonality of K_t and R_t. As results in Dixon (1976) show, when K_t and R_t are nondiagonal, there is considerable substitution between the controls. Intuitively, it might be expected that there would be no substitution with diagonal objective function matrices. This would occur if the A_t and B_t were square and diagonal. The substitution that does occur is as expected. In classes that have their target volume, some alteration is made in those controls and, hence, states so that the loss is spread among classes. For example, in S4, harvests are increased in the M10 in the initial period to counterbalance the decreases in the M42 and M30 classes. With a quadratic loss function, this is a predictable result.

A final aspect of the initial–period controls is the effect of uncertainty in A_t and B_t. The impact of uncertainty is registered in several aspects of the results. In this section attention is restricted to the initial–period response. In S6, which is identical to S1 except that the randomness of A_t and B_t is explicitly included in the solution, a comparison of the initial–period controls between S1 and S6 shows that the controls are almost identical. Thus, the effect of uncertainty about the coefficients in the dynamics on the control vector is slight. Table 4 also gives the initial–period results for S6 and S9. The norm of the control vector is 0.353 in S1 and 0.352 in S6. Likewise, the sum of deviations of control vector is 0.360 in S1 and 0.359 in S6. In S9, the stochastic counterpart to S4, the effect of uncertainty is more pronounced. The control norm is 4.83 in S4 and 4.49 in S9. In percentage terms there is a larger difference between the norms when the objective function weights δx_t more highly. The trend continues as the ratios of initial period norms for S2 to S7, S3 to S8, and S5 to S10 are 1.02, 1.03, and 1.10, respectively. It can be concluded that, as more emphasis is placed on timber stocks being at their predetermined levels, the effect of uncertainty in determining the optimum harvest levels becomes more influential. Phrasing this result another way, if a certainty–equivalent approach is taken, the error likely to be made in determining optimal harvest level will be greater when deviations from desired timber stocks are more important than harvest–level deviations.

The above results suggest that the assumptions made about uncertainty should depend on the objectives of management. When there is little concern about the state of the system, uncertainty in A_t and B_t has little effect on the control effort. When policy emphasis is shifted onto the timber stocks, then the same level of uncertainty has a larger effect on control actions. A plausible interpretation of this is that, when only control levels are important, the controls can be readjusted in each time period to give the desired flow. When the policy is expanded to include losses from stock deviations, then the controls must be used also to counterbalance the effect of the uncertain growth functions which now have an added importance.

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[1]For successive periods, it is difficult to judge the degree of substitution since frequently a class has a large deviation but no control actions are permitted. Thus, it is not possible to compare own–class and cross–class effects as clearly as in the initial period.

The desirability of active learning with respect to A_t and B_t cannot be fully assessed until the costs of uncertainty about A_t and B_t have been analyzed. This is done partially in 5.1.3 and completed in 5.3. In this latter section the necessity of employing active learning for the particular model used is considered.

5.1.2. Analysis of Harvest and Volume Trajectories Under a Certainty–Equivalent Specification

Having analyzed the response of the initial–period controls, attention is now focused on the behavior of the control and state vectors over the time horizon. The analysis is directed to determining the policy–relevant range of the value of K_t to R_t, *i.e.*, to determine when further increases in one set of values relative to the other will have negligible impact on control effort. Additionally, the control and state trajectories are examined to determine the variation from the deterministic results when the randomness in the A_t and B_t is explicitly incorporated into the solution.

Before discussing either the CE or stochastic (S) trajectories, the accuracy of the results can be assessed by noting that in S1–S10, 5 percent of the actual control levels are less than 50 percent of the target harvests and less than 6 percent of the controls are 50 percent larger than the target harvests. Since the analysis in 4.3.4 shows that the LQG model is highly accurate for the vast majority of the actual control levels, the linearized model is an accurate portrayal of the state of the forest for the indicated control actions. This, of course, is contingent upon the accuracy of the deterministic model.

Table 5 displays the harvest and stock trajectories for S1 and S2. Net stock and removal flows are given since those are usually of interest to a forest manager. To measure the response of the controls to deviations in the level of timber stocks, the Euclidean norm of the control vector is divided by the norm of the stock deviation vector, *i.e.*, the first 15 components of $\delta x_{t|t}$. This is an approximate measure of control effort.[1] Clearly, the less responsive the controls, the closer this ratio would be to zero. This ratio is not bounded above by one; but, in practice, values in excess of one are rare as indicated by all of the results in S1–S16. Using this ratio, the sensitivity of control actions to different policies can be revealed by comparing these ratios across simulations for corresponding time periods. Such a comparison is exact only when the state vectors are identical, component for component. Such is the case for only the initial–period controls.

The harvest trajectory of S1 given in Table 5 remains quite close to the target levels in early periods, largely ignoring deviations in the timber stocks. The modification in harvest and thinning levels in the early periods is slight in S1 compared with δu_0 under different management policies. This response is to be expected though since the objective function is weighted almost exclusively to maintaining harvest flows. The control vector begins to respond toward the end of the planning horizon when the model must balance the deviation in current harvests with deviations in the terminal timber stocks. The trajectories indicate that potential losses in current harvests are postponed to a less costly future. When there is little change in the control vector, the controls may become particularly susceptible to feasibility violations for the reason given in 5.1.1.

The pattern of harvests of S1 confirms that nondeclining yield is not a completely specified objective function in a stochastic environment. The preference structure in S1

[1] For further discussion of measuring control responsiveness, see Chow (1975).

TABLE 5

Comparison of Volume and Harvest Deviations Over Time by Period for Simulations One (S1) and Two (S2)

Period	S1			S2		
	Net volume deviations[a]	Net harvest adjustments[a]	Ratio of harvest to volume deviations[a]	Net volume deviations[a]	Net harvest adjustments[a]	Ratio of harvest to volume deviations[a]
	million cubic feet					
0	10.67	0.36	.042	10.67	1.81	.161
1	7.67	– 0.03	.005	6.05	3.10	.103
2	82.46	0.01	.004	77.37	4.55	.054
3	43.47	– 0.24	.008	33.27	0.53	.030
4	42.13	1.18	.014	30.73	5.94	.084
5	73.99	1.80	.018	56.72	5.72	.070
6	67.84	7.68	.143	45.38	6.78	.163
7	30.81	2.87	.054	8.13	1.11	.060
8	-27.98	3.84	.206	-49.69	0.56	.202
9	– 1.54	20.93	.610	-20.49	12.30	.487

[a]Positive numbers indicate that the numbers are below their target levels; all numbers are the quotient of the Euclidean norm (the square root of the inner product of a vector) of the control vector divided by the norm of the first 15 elements of the state vector, *i.e.*, the timber volumes.

Source: Computed.

is a plausible structure for a preference function representing nondeclining yield under uncertainty. Table 5 shows that, over the time horizon, harvest levels are below the target levels and, furthermore, that the net deviations are not in a nondeclining pattern. As argued earlier, no plan of positive harvest levels can be assured with a probability of one so that any level of nondeclining yield is associated with some level of probability that the planned harvests will not be met.

The results provide a further insight into the nature of the nondeclining–flow policy. The preference structure in S1 could be improved as a representation of nondeclining flow by using a *negative* discount rate. That would have the effect of making larger adjustments in the present and smaller adjustments in the future.[1] Zivnuska (1975) argues that nondeclining flow has the effect of bringing any future fall–offs in harvests into the present where, with a positive discount rate, they have the greatest cost. Furthermore, if the nondeclining objectives are altered to consider uncertainty, then high levels of confidence will require low levels of harvest since the policy–maker must consider future *possible* losses in timber. That effect will be lessened if a positive discount rate is used and amplified if a negative rate is used.

When the K_t are given small positive values, the results are surprising. In S2 the nonzero diagonal elements of K_t are set equal to 0.10. The change in control response is considerable. For example, the ratio of the initial–period control and state norms in S1 is 0.042, whereas the quotient increases by more than a factor of three in S2. Similar increases in the responsiveness of the removal flows can be observed by comparing the norm ratios in Table 5 (S1 and S2) for the first eight time periods. When the K_t values are increased again to 0.2, the factor of increase in responsiveness is much less from S2 to S3 than from S1 to S2. As this pattern is continued, incrementation of K_t relative to R_t has a diminishing effect on the responsiveness of the controls. This latter effect can be explained by examining the control law, G_t. As K_{t+1} grows larger relative to R_t, the effect of R_t is overwhelmed[2] so that G_t,

$$G_t = -(R_t + B_t' H_{t+1} B)^{-1} B_t' H_{t+1} A_t,$$

increasingly ignores R_t. Further increases in K_{t+1} do little to change G_t since both H_{t+1} and its inverse are in G_t.

Table 6 further illustrates this effect by displaying the ratio of control to state norms for S1–S10. In most periods this ratio increases, but the rate of increase declines as K_t becomes larger. Thus, the region of substantial trade–offs, in terms of control effort, is where the elements of K_t are less than those of R_t. Once the K_t become larger than R_t, the effect of further increases is relatively insignificant.

The reason for this particular range of sensitivity is clear when the economic impact of the weights, K_t, are considered. When a forest exists solely as a source of timber for harvest, a unit of timber has a product value in the sense of yielding a revenue in the

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[1]The results in Howitt, Dixon, and O'Regan (1977) show that, with a zero discount rate and preferences similar to those in S1, the initial period response increases substantially compared to the response in S1.

[2]The recursive effect of K on H can be seen in equation (34), Section 3, *supra*, p. 12.

last period of its existence. When the K_t become positive, it implies that the timber stocks produce a direct benefit (utility) in each period of their existence. Clearly, if an owner is receiving nearly the harvest value of a tree in each period, the maximizing behavior is to let the tree stand. When the multiple–use value is ascribed to the forest and quantified in units comparable to the price of harvested timber, the value maximizing rotation can be quite different from what has been traditionally considered in the forestry literature.[1]

TABLE 6

Ratio of Harvest Deviation Norms to Volume Deviation Norms by Period for Simulations One (S1) Through Ten (S10)

Period	Simulation									
	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
0	.042	.160	.252	.579	.806	.042	.158	.244	.539	.730
1	.005	.103	.161	.328	.436	.005	.101	.156	.312	.431
2	.004	.054	.080	.126	.132	.004	.053	.078	.126	.143
3	.008	.030	.038	.141	.375	.008	.030	.036	.106	.250
4	.014	.084	.132	.248	.312	.015	.079	.119	.238	.321
5	.018	.070	.100	.158	.160	.020	.070	.098	.155	.171
6	.143	.163	.178	.327	.513	.144	.160	.171	.272	.410
7	.054	.060	.063	.686	.140	.054	.055	.055	.058	.087
8	.206	.202	.192	.214	.252	.199	.194	.191	.200	.228
9	.610	.487	.380	.193	.107	.602	.500	.414	.225	.153

Source: Computed.

The impact of K_t becoming nonzero is also evident in the numerical values of the objective functions. The term $\delta x'_{0|0}\ H_0\ \delta x_{0|0}$ is the present value cost of the current deviations and is shown later. In S1 this cost is 2.62. In S2 the cost for the identical $\delta x_{0|0}$ is 17.46. For S3, which doubles the K_t in S2, the cost is 30.30––a much less significant jump in terms of percent than that from S1 to S2. This behavior is continued in S2–S5. For example, the deviation cost is 107.32 for S4 and 331.25 for S5. Thus, slight perturbations of K_t around zero, given that the R_t values are unity, will have the largest percentage impact on relative costs. Therefore, it is clear that precision in knowing society's goals for national forest management is most critical when losses from stock deviations are less than losses for equivalent deviations in harvest levels.

5.1.3. The Effect of Random System Parameters on Harvest and Volume Trajectories

Random A_t and B_t have less clear effects on the harvest and stock trajectories than on the initial–period controls. In comparing S1 with its stochastic counterpart, S6, the control responses in S6 appear to be slightly more aggressive than the certainty responses.

[1]For a more detailed discussion of this point, see Dixon and Howitt (1977).

That is ascertained by comparing Tables 5(S1) and 7(S6) and by observing that in identical periods the ratio of norms is greater in three periods and lesser in only two periods. In general, however, the effect of uncertain A_t and B_t is not significant in the sense that the levels of stock and harvest deviations are almost identical when comparing the levels between periods.

When values are assigned to the stocks, there is a detectable though not substantial change in the pattern. A comparison of the period–by–period harvest and stock deviations for S4 and its stochastic counterpart, S9, shows that the absolute value of total net stock deviations are greater in the stochastic model than in the certainty–equivalent model except for the initial period as shown in Table 7(S4) and (S9). Further, the ratios of norms in S4 are greater than the corresponding norms in S9 in 8 out of 10 periods. This pattern suggests that, as more weight is put on having the desired stock levels, the optimal behavior is to become more conservative in the control response. This conservatism translates to staying closer to predetermined removal levels although, in all of the comparisons between simulations, the harvest– and stock–deviation patterns remain quite similar. That is not surprising since the relative level of uncertainty is moderate in terms of statistical significance.

Recall that, in comparing the trajectory of stochastic norms between S1 and S6, the stochastic policy tended to be slightly more responsive. In comparing the corresponding period–by–period norms of S2–S5 with their stochastic counterparts (S6–S10), as can be done utilizing Table 6, the stochastic norm ratios are less than the corresponding certainty–norm ratios in three–fourths of the periods. However, the differences between these ratios do not always grow as K_t increases relative to R_t. Given the difference in norm behavior between S1 and S6 and between policies which attribute losses to stock deviations, the effect of linear–coefficient uncertainty varies with the management policy.

The effect of the random A_t and B_t on the cost of current stock deviations, $\delta x'_{0|0}\ H_0\ \delta x_{0|0}$, varies with the objective function. The cost due to the initial deviation is 2.67 in S1 and 2.69 in S6, an increase of roughly 0.7 percent over S1. When K_t increases from 0 to 0.1, costs are greater by .5 percent in S7 than in S2. As shown in Table 8, the change in the percentage of costs is related to K_t, with the increase in percentage growing at a declining rate with respect to K_t for the larger K_t values. This indicates that the current deviation cost due to uncertainty in A_t and B_t usually increases with a shift in emphasis toward the state deviations but that, once the preponderant emphasis is on the states, further increases in K_t do not lead to substantially increasing rates of this cost due to uncertainty about A_t and B_t.

Even though uncertainty about A_t and B_t does not have a substantial impact on harvest actions or current deviation costs, active learning with respect to A_t and B_t cannot be ruled out categorically. The impact of uncertainty about A_t and B_t on the cost of uncertainty terms in the objective function (37) must also be examined. This is done in 5.3, and it is then possible to consider the necessity of various forms of active learning more fully for the empirical model of this study.

5.2. Optimal Stock Estimates: The Kalman Filter

The Kalman filter offers a distinct departure from traditional mensurational techniques. Since the filter appears to have substantial potential for timber management models, its operation on a particular state variable is examined.

TABLE 7

Comparison of Volume and Harvest Deviations Over Time by Period for Simulations Four (S4), Six (S6), and Nine (S9)

Period	S4			S6			S9		
	Net volume deviations[a]	Net harvest adjustments[a]	Ratio of harvest to volume deviations[a]	Net volume deviations[a]	Net harvest adjustments[a]	Ratio of harvest to volume deviations[a]	Net volume deviations[a]	Net harvest adjustments[a]	Ratio of harvest to volume deviations[a]
	million cubic feet								
0	10.67	6.36	.579	10.67	0.36	.042	10.67	5.90	.539
1	0.93	9.90	.328	7.67	- 0.02	.005	1.44	9.01	.312
2	64.55	10.81	.126	82.45	0.06	.004	66.06	10.78	.126
3	13.03	- 3.31	.141	43.40	- 0.27	.008	14.65	- 2.48	.106
4	13.83	11.17	.248	42.09	1.23	.015	14.65	11.03	.238
5	33.15	9.02	.158	73.90	1.94	.020	34.16	9.14	.155
6	16.90	- 1.62	.327	67.60	7.94	.144	17.87	- 0.63	.272
7	-13.03	- 4.30	.086	30.26	3.07	.054	-13.23	- 3.03	.058
8	-65.56	- 3.28	.214	-28.84	3.50	.199	-67.57	- 4.10	.200
9	-32.56	2.50	.193	- 2.04	19.99	.602	-33.93	2.45	.225

[a]Positive numbers indicate that the numbers are below their target levels; all numbers are the quotient of the Euclidean norm (the square root of the inner product of a vector) of the control vector divided by the norm of the first 15 elements of the state vector, *i.e.*, the timber volumes.

Source: Computed.

TABLE 8

Comparison of Cost of Initial Deviations
Between Certainty-Equivalent
and Stochastic Models

Simulation	Certainty-equivalent, $\delta x'_{0\|0}\ H_0\ \delta x_{0\|0}$	Stochastic, $\delta x'_{0\|0}\ H_0\ \delta x_{0\|0}$	Ratio[a]
	1	2	3
S1 S6	2.67	2.69	1.007
S2 S7	17.46	17.55	1.005
S3 S8	30.30	30.56	1.009
S4 S9	107.32	111.88	1.042
S5 S10	331.25	363.70	1.098

[a]Column 2 divided by column 1.

Source: Computed.

Selected for the purpose of exposition is timber class M42 (young saw timber). In the initial period of S4, the optimal solution is to decrease regeneration harvesting by 4.675 million cubic feet from its target level. In the next period, therefore, the equations of motion predict that the deviation in this volume class will be equal to 3.231. To illustrate, let δVol_1 denote the volume deviation of timber in period 1 for M42 and let δBA_0 be the deviation in basal area of the M42 class in the initial period so that

$$\delta Vol_{1|0} = .049\ \delta BA_{0|0} - 1.143\ \delta u_0$$
$$3.231 = .049 \cdot 175 - 1.143 \cdot 4.675.$$

The variance of this prediction is 82.99. The sampling of basal area in the M42 class at the beginning of the next period indicates a deviation in volume of .36 million cubic feet. The variance of this sampled estimate in this class is 76.96, so there is considerable uncertainty surrounding the estimate of volume based on the current sample of basal area.

The Kalman filter then combines the prediction based on prior observations, $\delta Vol_{1|0}$, with the estimate based on the most current sampling, δz_1, using (31). Essentially (31) combines two weighted estimates of δVol_1 so that, when added together, the best unbiased estimate of δVol_1 is given as $\delta Vol_{1|1}$ = 1.884. This estimate lies between the two values that are given as estimates of δVol_1. The variance of this combined estimate is 39.95. Thus, instead of having an estimate with a variance of 76.96, which relies solely on current samples, use of prior samples reduces the variance by more than 48 percent. It is noted that less reliance probably would be placed on $\delta Vol_{1|0}$ if the stochastic nature of A_t and B_t were recognized which it is not by the Kalman filter. When that source of uncertainty is explicitly recognized, the uncertainty of the predicted estimate, $\delta x_{t|t-1}$, would most likely increase, placing relatively more reliance on the observation (sampling) procedure.

In general, the impact of the Kalman filter is to decrease the variance of the estimated stock variable levels by about 50 percent from the observer variance. This is shown in Table 9 which displays the observer variances for the six wild stands and their corresponding Kalman estimate covariances for the first five periods for S1–S10. These results indicate very clearly that use of recursive estimation results in a substantial decrease in uncertainty, given the magnitudes of the parameters in this problem. The decrease in uncertainty in any given application is, of course, a function of the relative sizes of A_t, C_t, Ω_t, and θ_t.

To observe the effect of a different sampling precision, a simulation was run reducing all of the elements in θ_t by 75 percent. This effectively reduces the size of the observation standard error by 50 percent. Thus, more reliance is placed on the sampled estimates and less on the predict, $\delta x_{t|t-1}$. The control in the initial period is still 4.675, and the predicted deviation in volume is still 3.231. The sampled value δz_1 is $-.05$; but the conditional updated estimate $\delta x_{t|t}$ is .079 which is closer to the sampled value than with the less precise observer, and the variance of the new estimate is 15.62. Intuitively, this is what would be expected.

Experiments such as the above can determine the effect of greater precision, but it must be recalled that the cost of this greater precision is not part of the control problem so that optimal sampling intensity is not a result of the model.

The Kalman estimation procedure essentially introduces a new step into traditional survey techniques: the intermediate prediction, *i.e.*, the prediction based on all past sample information and the estimated growth dynamics. It has been assumed in this study that plots have been selected by a simple random sample. Clearly, the sampling strategy could be altered to suit the specific estimation goals. For example, if the current objective is to obtain the best possible estimate of basal area in condition classes, the sampling strategy used might be different from that if the objective were to estimate timber volume. If the objective were to estimate the coefficients of A_t and B_t, then another sampling technique might be better. Choosing the "best" sampling plan is intimately related with a topic in stochastic control known as identifiability (Aoki, 1967).[1] Basically, if a system is identifiable, then—with a sufficient number of observer signals—it is possible to estimate A_t and B_t. In any of these cases, a recursive estimator, of which the Kalman filter is a special case, generally will be used to estimate the current state variables.

[1]The engineering definition of identifiability is not to be confused with the identifiability problem in econometrics although they are closely related.

TABLE 9

Comparison of the Observed Variances and Updated Variances of Timber Classes by Period for Simulations One (S1) Through Ten (S10)

Period	Timber class[a]											
	M01		M10		M20		M30		M41		M42	
	Observed	Updated	Observed	Updated	Observed	Updated	Observed	Updated	Observed	Updated	Observed	Updated
1	0.03	0.02	120.0	61.2	1435.0	708.0	5403.0	2700.0	1456.0	716.0	77.0	39.6
2	0.03	0.01	139.0	68.1	1567.0	746.0	2378.0	1187.0	1705.0	816.0	80.9	41.1
3	0.03	0.01	158.0	76.2	1598.0	755.0	579.0	302.0	1868.0	884.0	84.9	43.1
4	0.00	0.00	177.0	84.5	332.0	168.0	589.0	283.0	1916.0	905.0	89.0	45.1
5	0.00	0.00	196.0	92.7	338.0	162.0	584.0	277.0	1096.0	526.0	93.2	47.2

[a]The timber classes M01 to M41 represent wild stands.

Source: Computed.

5.3. The Costs of Uncertainty

One of the major sources of information generated by use of the LQG technique is an evaluation of the costs of various sources of uncertainty in a control model. This section investigates the impact of the various sources of uncertainty and differing policies. In addition, the impact of lowering various sources of uncertainty is measured. The purpose of analyzing uncertainty is to determine where research or sampling efforts should be directed to lessen uncertainty and increase expected returns.

Table 10 gives decompositions of the expected value of several of the simulations for the 10 planning periods. The reader is reminded that these costs are invariant under a linear transformation of K_t, K_T, and R_t. In Table 10(S1), the most striking result is that the fourth column is always greater than the fifth column on a period–by–period basis. This means that the cost of uncertainty in future growth dynamics, $\Sigma \mathrm{tr}\ H_{t+1}\ \Omega_t$, is greater than the future costs of uncertainty in estimating the state vector. At the beginning of a time period, steps can be taken to lower Ω_i, $i \geqslant t$ and θ_i, $i > t$.[1] The results for S1 show that more is to be gained by decreasing uncertainty in Ω_i than θ_{i+1}. Anytime Ω_i is lowered, it has a two–pronged effect on costs. First, tr $H_{i+1}\ \Omega_i$ will be directly reduced. The second effect is on $P_{i+1|i+1}$. Recall from (28) and (30) that the magnitude of $P_{i+1|i+1}$ is related to the magnitude of Ω_i. Thus, reducing Ω_i will effect reductions in costs for terms in the objective function containing $P_{i+1|i+1}$. The impact of the Ω_i in S1 can be measured in another way. $\Sigma \mathrm{tr}\ H_{i+1}\ \Omega_i$ accounts for 65 percent of the expected loss in $E(J_0)$. This percentage is a lower bound on the cost of Ω_i since it does not account for the effect of Ω_{i-1} on $P_{i|i}$.

When the policy ascribes a loss to not having desired timber stocks, almost all of the costs increase substantially. Additionally, the levels of corresponding cost of uncertainty terms in S2–S5 are approximately proportional to the changes in the K_t. When the K_t assume nonnull values, they account for roughly 99 percent of the total initial period expected costs in S2–S5, whereas they only account for 77 percent of the initial period total costs in S1.[2] Thus, when having stocks at a specified level is important, the relative cost of a given level of stock deviations declines substantially. This result implies that it might be more advantageous to allocate budget from harvest activities to reducing uncertainty when stock deviations are costly. That is, reducing harvests would likely result in increased stock deviation costs, but a decrement in uncertainty will have a much higher payoff than when stocks do not have a temporal existence value.

In S2–S5 the cost of future growth–dynamics uncertainty, once again, is greater than the costs of future uncertainty in estimating the state vector. However, the cost of current uncertainty in the state vector, tr $H_t\ P_{t|t}$ accounts for 58 percent to 60 percent of all initial period costs in S2–S5 implying that, when stocks yield temporal existence values, it is important to be able to observe them precisely. This result seems to say that initial stock levels should be sampled more intensely. Given the strict structure of the problem, this is not an alternative. The initial covariance, $P_{0|0}$, is a given parameter; only θ_i, $i \geqslant 1$ can be altered. Stepping outside the strict structure of the model, the

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[1]In the context of the model, at the beginning of t, δz_t is observed so that θ_t cannot be changed.

[2]Since K_T remains fixed, tr $H_{10}\ \Omega_9$ and tr $A_9'\ H_{10}\ B_9\ G_9\ P_{9|9}$ remain the same in S1 through S5.

cost of initial uncertainty in the state estimate could be significantly reduced by use of a "prefiltering" cycle. This process begins with an initial state estimate at $t = -1$. Then, using (27) to calculate an update estimate, a sample is taken at $t = 0$ and combined with $\delta x_{0|-1}$. Then $P_{0|0}$ most likely would be considerably reduced over what it is now. In more concrete terms of a timber application, this requires using the data of a current and prior decennial survey and a predict relationship such as (27) to forecast what the current volumes are. This procedure would extract more value out of old sample data.

In determining whether resources should be devoted to sampling or reducing growth–dynamics uncertainty, the potential usefulness of the uncertainty reduction must be considered. Given the way that growth is modeled in the Stanislaus National Forest, more intense sampling will be likely to have only a small return. This is shown by comparing total expected costs of S1 with S11 and S4 with S12. The values of the expected costs for S11 and S12 are given in Table 11. In S11 and S12, all the θ_t's are reduced by 75 percent. In S11 the greatest drop in total cost in any of the periods is less than 10 percent of the corresponding costs in S1. A somewhat larger impact occurs between S4 and S12; in one period, total costs drop by about 30 percent.[1] However, there is not a perceptible difference in total initial period cost in either of the comparisons. The reason is that, in the variance of any update estimate, $P_{t+1|t}$, the vast majority of variance, as it pertains to cost, comes from the first 15 diagonal elements of Ω_t. Even if the estimate $P_{t|t}$ is very precise, substantial uncertainty will be introduced in the next period via the addition of Ω_{t+1}, and a very precise sample will again have to be taken to reduce $P_{t+1|t+1}$. If, however, the research effort were reversed and the Ω_i were reduced, which means that relationship (42) would be estimated more precisely, then all three of the cost terms would be reduced, perhaps substantially. Assume that a relatively precise estimate of $\delta x_{0|0}$ is available. Then with precise Ω_i, $i > 1$, all future estimates of $\delta x_{t|t}$ will be more precise since some of the precision of the former estimates are preserved over the time horizon.

Determining the cost of uncertainty is only one side of the problem. The other side of the problem is the cost of obtaining greater certainty. As noted earlier, that is an aspect of the management problem that the LQG problem does not consider. Even though future uncertainty in the system dynamics costs more than future observation error, it cannot be definitely stated that research effort should be devoted to Ω_i instead of θ_i. Such a decision must also consider the prospective returns from research.

The question of where research effort should be directed in the Stanislaus National Forest model can be shown to point clearly to relationship (42), relating volume of timber to basal area. That relationship affects both Ω_t and θ_t. In S13 and S14, θ_t is specified so that the basal area variances and covariances are reduced by 75 percent. S13 and S14 are respectively identical in preference structure to S1 and S4. Table 10(S4) and (S14) gives the decomposition for the objective functions. On total initial costs, the effect of the increased observer precision is slight. The objective function of S1 is less than 1 percent higher than S13 and that of S4 is not perceptibly different from S14. The costs of H_{t+1} Ω_t remain the same. In both comparisons the cost of current deviation uncertainty goes down substantially in S13 compared to S1 for all periods but the initial period although, from S4 to S14, the cost declines less than 5 percent in each period after the initial period.

[1]The fact that some of the expected costs in S11 and S12 are greater than their corresponding costs is due to the fact that a change in θ_t results in a change in δz_t. Thus, the state estimates and controls change which results in some of the state deviation costs being greater in S11 and S12.

TABLE 10

Decomposition of the Objective Function for Simulations One (S1), Four (S4), Fourteen (S14), and Sixteen (S16)[a]

Period (t)	Objective function, $E(J_t)$ 1	Current deviation cost, $\delta x'_{t\|t} H_t \delta x_{t\|t}$ 2	Current volume estimate uncertainty, $\mathrm{tr}\, H_t P_{t\|t}$ 3	Cost of growth dynamics uncertainty, $\mathrm{tr}\, H_{t+1} \Omega_1$ 4	Cost of future uncertainty in volume estimates, $\mathrm{tr}\, A_t H_{t+1} B_t G_t P_{t\|t}$ 5
			S1		
0	.117E + 02	.267E + 01	.132E + 01	.251E + 00	.141E - 01
1	.124E + 02	.445E + 01	.556E + 00	.218E + 00	.224E - 02
2	.140E + 02	.646E + 01	.354E + 00	.200E + 00	.277E - 02
3	.119E + 02	.461E + 01	.266E + 00	.153E + 00	.641E - 02
4	.121E + 02	.517E + 01	.179E + 00	.131E + 00	.373E - 02
5	.121E + 02	.525E + 01	.150E + 00	.174E + 01	.441E - 02
6	.753E + 01	.163E + 01	.950E + 00	.484E - 01	.940E - 02
7	.598E + 01	.986E + 00	.968E - 01	.147E + 01	.209E - 01
8	.514E + 01	.946E + 00	.784E + 00	.264E - 01	.169E - 01
9	.410E + 01	.668E + 00	.646E - 01	.334E + 01	.254E - 01
			S4		
0	.169E + 05	.107E + 03	.994E + 04	.409E + 04	.290E + 03
1	.498E + 04	.275E + 03	.222E + 04	.145E + 04	.633E + 02
2	.269E + 04	.953E + 03	.765E + 03	.543E + 03	.216E + 02
3	.794E + 03	.102E + 03	.283E + 03	.210E + 03	.606E + 01
4	.386E + 03	.837E + 02	.109E + 03	.883E + 02	.238E + 01
5	.255E + 03	.106E + 03	.458E + 02	.504E + 02	.948E + 00
6	.874E + 02	.102E + 02	.259E + 02	.266E + 02	.487E + 00
7	.504E + 02	.125E + 02	.137E + 02	.144E + 02	.252E + 00
8	.241E + 02	.721E + 01	.737E + 01	.607E + 01	.995E - 01
9	.781E + 01	.133E + 01	.311E + 01	.334E + 01	.254E - 01

	S14				
0	.169E + 05	.107E + 03	.994E + 04	.409E + 04	.290E + 03
1	.490E + 04	.310E + 03	.216E + 04	.145E + 04	.228E + 02
2	.273E + 04	.104E + 04	.737E + 03	.543E + 03	.707E + 01
3	.760E + 03	.822E + 02	.275E + 03	.210E + 03	.217E + 01
4	.377E + 03	.802E + 02	.106E + 03	.883E + 02	.862E + 00
5	.254E + 03	.108E + 03	.445E + 02	.504E + 02	.333E + 00
6	.852E + 02	.908E + 01	.254E + 02	.266E + 02	.182E + 00
7	.497E + 02	.123E + 02	.134E + 02	.144E + 02	.979E − 01
8	.245E + 02	.778E + 01	.726E + 01	.607E + 01	.380E − 01
9	.780E + 01	.138E + 01	.307E + 01	.334E + 01	.977E − 02
	S16				
0	.168E + 05	.107E + 03	.994E + 04	.404E + 04	.290E + 03
1	.483E + 04	.286E + 03	.215E + 04	.143E + 04	.226E + 02
2	.270E + 04	.103E + 04	.730E + 03	.537E + 03	.638E + 01
3	.752E + 03	.840E + 02	.272E + 03	.207E + 03	.168E + 01
4	.371E + 03	.789E + 02	.105E + 03	.871E + 02	.635E + 00
5	.253E + 03	.109E + 03	.439E + 02	.498E + 02	.249E + 00
6	.825E + 02	.732E + 01	.251E + 02	.263E + 02	.126E + 00
7	.485E + 02	.116E + 02	.132E + 02	.142E + 02	.650E − 01
8	.231E + 02	.658E + 01	.716E + 01	.601E + 01	.255E − 01
9	.756E + 01	.121E + 01	.302E + 01	.332E + 01	.651E − 02

[a] The sum of columns 2–5 in any one period does not equal the value in column 1. Note that J_t also includes the sum of the costs in the remaining periods in the last two columns as in equation 37, Section 3, *supra*, p. 19.

Source: Computed.

In S15 and S16 the last 18 diagonal elements of Ω_t and its covariances are also reduced by 75 percent as well as basal area observation precision being similarly reduced. Comparing S4 with the decomposition of costs of S16 given in Table 10(S16), there is less than a 1 percent drop in total initial period costs. Thus, the bulk of uncertainty cost, which is by far the larger cost in all of the simulations, is directly attributable to uncertainty arising from the relationship of basal area to volume.

TABLE 11

Expected Total Costs by Period for
Simulations Eleven (S11) and Twelve (S12)

Period (t)	Total cost, $E(J_t)$ S11	S12
0	.116E + 02	.169E + 05
1	.112E + 02	.358E + 04
2	.138E + 02	.372E + 04
3	.114E + 02	.802E + 03
4	.121E + 02	.319E + 03
5	.122E + 02	.263E + 03
6	.727E + 01	.766E + 02
7	.583E + 01	.496E + 02
8	.483E + 01	.255E + 02
9	.397E + 01	.766E + 01

Source: Computed.

In discussing the various sources of uncertainty, it has been implicitly assumed that the various uncertainty sources are estimated from different data bases. If the approach is taken that δz_t, δu_t, $\delta x_{0|0}$, and θ_t are the only information the decision-maker has and that all of the values for A_t, B_t, Ω_t are uncertain, then the above analysis must be modified. If δz_t is used to estimate the parameters, A_t, B_t, and Ω_t, as well as δx_t, which it would be with a conventional econometric systems estimator, then the precision with which δz_t is observed affects the precision of all the other parameters. Of course, such a problem involves active learning which is not addressed empirically in this study.

When A_t and B_t are random, most costs affected by the random, A_t and B_t, rise. In comparing S1 with S6, the initial period expected total costs rise by about 4 percent and most dramatically for H_{t+1} Ω_t, particularly in the early periods. This trend is not evident when the K_t take on nonnegative values. In comparing S2–S5 with their respective stochastic counterparts, the costs tend to be quite similar except for the costs of future state estimation uncertainty.[1] For example, $E(J_0)$ is 1730.0 in both S2 and S7. This illustrates the differing influence of uncertainty in these simulations. Stochastic A_t and B_t make for very little difference in control actions for preferences that do not emphasize harvest flows, but their randomness does noticeably increase the costs of uncertainty. That trend is reversed when K_t becomes positive.

It is now possible to address the question of whether active learning is justified for this problem. Considering first the problem of uncertainty in the A_t and B_t, the empirical results show that both controls and costs are essentially little affected by this source of uncertainty. The effect of the linear coefficient uncertainty varies with the particular policy; but overall, active learning would not significantly reduce costs. However, substantial costs have been attributed to the uncertainty represented by the additive error term in the volume to basal area regression equation. An active learning solution with regard to this source of uncertainty would yield more precise estimates of the mean and variance of the additive error, but there is no reason to believe that a more accurate estimate of the variance of this error would be smaller than the current estimates. Thus, active learning with respect to additive uncertainty will not necessarily yield lower overall expected costs.

As discussed in 3.2.1, active learning for LQG problems presents some extraordinarily difficult problems. Most research examining active learning has assumed that the A_t, B_t, and C_t are unknown constants and not partial derivatives from a Taylor–series expansion. A very pragmatic alternative to active learning in this application is to reexamine the specification of (48). More independent variables could be introduced as well as using a different functional form. Either of these approaches could reduce the costs of uncertainty and, in terms of practicality, could be directly applied. The whole topic of active learning in LQG methodology requires considerable theoretical development.

The following conclusions can be drawn. First, uncertainty regarding future–growth dynamics in the Stanislaus National Forest is currently more costly to achieving systems goals than the future–observer (sampling) system. Of course, as the size of Ω_t to θ_{t+1} changes, this result may change. Second, sampling more intensely for basal area will deduct little from the cost of uncertainty. Third, ascribing a loss to volumes substantially increases all costs of uncertainty and makes uncertainty costs a larger proportion of initial–period expected total costs. Fourth, the model provides a means whereby the cost reductions from lessening uncertainty can be measured in terms of the system's objective function.

5.4. Summary

This section has compared and examined results from 16 simulations. The control actions are in accordance with intuitive expectations. For minimizing the loss from timber–flow deviations, losses are avoided in the present and near future, being deferred to the less costly distant future. By varying preference weights, the controls were shown to be sensitive to different policy intensities. Including a positive temporal existence value for timber stocks changes the harvest flows from those that assign a zero temporal existence value. The extent of the disparity will depend on the size of the weights on the stocks relative to the weights on timber flows.

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[1] These costs are a relatively small component of the total costs.

It was determined that the sensitivity of the controls to deviations in the state variables is a function of the size of K_t compared with R_t. As one matrix of weights becomes much larger than the other, the control response is insensitive to a greater disparity. The assumption that A_t and B_t are constants appears justified in the LQG model estimated since including the variances of the A_t and B_t does not have a substantial impact on costs or controls. This behavior is consistent with Prescott's (1972) finding that, when parameter uncertainty is low (for A_t and B_t), the certainty–equivalent solution is a good approximation to the active learning solution.

The experimental results show that feasibility constraints may exclude applying the LQG method to some resource problems. For variables that lie on the boundary of their feasibility set, infeasibilities can frequently occur in the LQG controls. That is, when a variable takes on a boundary value in the deterministic model, it can be adjusted in only one direction. The LQG does not recognize such a constraint. For variables that do not lie on a boundary, feasibility problems are usually avoided in the above simulations because the control rule usually indicates feasible movements about the target value. Constructing B_t so that variables which lie on a feasibility boundary cannot be readjusted avoids the problem of infeasible variable levels though there is some loss of flexibility in the solution. Thus, the LQG model can be applied to at least some microlevel problems.

In addition to solving the control problem, estimation of the inaccessible states is done in a way that is optimal with respect to the control problem. This results in a large decrease in the variances of the state estimates. Because of the recursive nature of the Kalman filter the information value of observations is increased over static estimation techniques. This is important considering the cost of sampling.

Another facet of the estimation and control complementarities is that the costs of uncertainty about the actual system dynamics as well as the observation process are determined. Regardless of the cardinal values computed, these costs are comparable in terms of which source accounts for a given percentage of total costs of uncertainty. The results indicate that the greatest benefit in reduced uncertainty from research in the Stanislaus National Forest would be derived from a more precise relationship between basal area and timber volume.

6. SUMMARY

The empirical planning of production from natural resources is an intertemporal stochastic optimization problem. For all but the simplest functional forms of such problems, computational requirements leave no other option than approximate optimization techniques. The LQG technique gives a stochastic approximate solution to the underlying problem although, in its simpler forms, the LQG neglects the active learning aspects of stochastic optimization.

In many respects the LQG model is well suited for modeling production from natural resources. The approximate model is also an intertemporal model so that the resulting LQG controls consider user costs directly in terms of future and current deviations. Benefits or costs that accrue to stock levels as a result of temporal existence values can be directly accounted for in the LQG objective function. The costs of various sources of uncertainty are quantified as a function of the particular policy employed. The problem of stock estimation is considered simultaneously with the control problem in the LQG model. This is a particularly important feature for forestry models since harvest scheduling and sampling problems in forestry are generally intertwined. Even when the problems separate, the optimal estimator for the total problem is a recursive estimator and not a series of samples over time that successively ignore all past sample data.

The major drawback of applying the LQG technique to natural resource problems is its inability to handle feasibility constraints optimally. Feasibility problems can become a significant difficulty if the target levels of variables lie on a feasibility boundary and the perturbations of the actual variable levels are large. In the present study, variables lying on a feasibility boundary are constrained to that boundary. Future research efforts should be directed toward developing methods that handle this problem more efficiently.

An additional area of needed research is determination of how approximate the LQG technique is. This will be particularly difficult on two counts. The first problem is that such a test requires the optimal stochastic solution to be computed and compared with the LQG solution—a difficulty discussed earlier. The second problem is how much the reliability of the LQG solutions varies with the functional form and degree of uncertainty in the underlying problem. Barring a major methodological breakthrough, such testing must await a generation of computers having increased capacity.

The particular harvest–scheduling model selected for analysis is transformed into the required LQG functional forms with varying degrees of success. Current Forest Service policies cannot be represented as a scalar–valued functional. As an alternative, a set of objective functions is used that partially spans the possible policies representable by a quadratic form. Estimation of the linearized dynamics is straightforward but results in a loss of information *vis–à–vis* that of the Timber RAM. In so doing, however, it provides a degree of flexibility that RAM lacks.

The experimental runs reveal that the policy trade–offs between having desired timber stocks and harvest flows become sensitive when the cost of stock deviations is less than the cost of equivalent harvest deviations. Alternatively, when there is uncertainty as to what the societal preferences are, precise estimates are most valuable when the loss for stock deviations is less than harvest deviations. Once the cost of stock deviations surpasses the cost of harvest deviations, little change is noticed in either control actions or relative costs of uncertainty. In general, the impact of uncertain A_t and B_t is slight, but it varies with the particular policy. For harvest–oriented policies, the impact of uncertain A_t and B_t is slight on control actions but more pronounced on uncertainty costs. Effects are opposite when harvest deviations take on loss values.

Given the low level of uncertainty surrounding the A_t and B_t parameters, the Kalman filter is a good approximate estimator for the timber volumes. The results show that, for the estimation procedure employed on the Stanislaus National Forest, the variance of the volume estimates is effectively reduced by about 50 percent over what a nonrecursive estimator would yield. That is a particularly important finding since data collection is not costless.

Even though the precision of the A_t and B_t estimates is high for a majority of the parameters and the inclusion of their variances in the solution does not have a dramatic impact on costs or control actions, considerable uncertainty exists in the model. From a cost point of view, the uncertainty surrounding the error term on the volume–to–basal area relationship is considerable. This large cost calls for at least a reexamination and, perhaps, respecification of this relationship.

One of the more uncomfortable aspects of this study is the seemingly overgenerous use of "it is assumed" or "assuming that." Most of those uses could be eliminated by a single grand assumption that uncertainty does not exist. Prior studies have done that, but one of the underlying themes of this study has been to see how the problem changes when uncertainty is recognized. The results show that uncertainty does have a significant impact on both harvest levels and returns to enterprise. Thus, in the presence of what may seem to be moderate or benign sources of uncertainty, stochastic optimization may be of considerable assistance.

APPENDIX

THE MODEL PARAMETERS

The parameters for all of the matrices used in determining the numerical results are listed. To avoid unnecessary repetition, the parameters are given in a concise manner. In order, they are the system dynamics, the covariance matrices, and the target levels for the state and control targets. Since all of the parameters are time varying, they are given in order of period.

In the following Section 1, the elements of A_t and B_t are listed in the same form that would be read by a FORTRAN program. Specifically, only 32 elements of the A_t can be nonzero and no more than 48 elements of the B_t where $B_t = A_t\ B1_t + B2_t$. At the beginning of the list of A_t parameters, two sets of indices are listed. The first gives the row designation of the element of A_t and the second the column designation. The elements of A_t are then listed. For example, the 16th number in any period listing corresponds to the 16th pair of indices. That is, the 16th row index is 17, and the 16th column index is 16 so that in period 0 the (17, 16) element of A is 0.0. In period 1 the (17, 16) element is 1.6500. A similar method is used to list the elements of B_t. The variances of the A parameters are listed after all of the coefficients over the time horizon have been listed in exactly the same order as the nonzero elements of A. A similar scheme is used for the variances of the B coefficients.

In Section 1 the covariance matrices $P_{0|0}$, Ω_t, and θ_t are given. The diagonal elements of those matrices are given first and then the covariances for the respective periods. $P_{0|0}$ is referred to in Section 1 as the initial covariance of the state vector. In Section 3 the target values for use and x_t are listed by time period. They are listed across by rows as are the other parameters.

1. THE EQUATIONS OF MOTION COEFFICIENTS: MEANS AND VARIANCES

THE A MATRIX

ROW DESIGNATIONS

1 2 3 4 5 6 7 8 9 10 11
12 13 14 15 17 18 19 20 21 22 23
24 25 26 27 28 29 30 31 32 33

COLUMN DESIGNATIONS

18 19 20 21 22 23 24 25 26 28 29
30 31 32 33 16 17 18 19 20 21 22
23 24 25 26 28 29 30 31 32 33

PERIOD = 0

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	.0415	.0375	.0395
.0401	.0361	.0493	0.0000	1.5391	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0275	1.1065	1.0534	1.0416
1.1131	1.0837				

PERIOD = 1

.0263	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	.0421	.0383	.0401
.0408	.0369	.0497	1.6500	0.0000	1.4577
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0217	1.0874	1.0416	1.0341
1.0903	1.0782				

PERIOD = 2

0.0000	.0292	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	.0428	.0390	.0408
.0415	.0378	.0501	1.6500	1.5391	0.0000
1.3476	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	1.0178	1.0709	1.0341	1.0275
1.0730	1.0728				

PERIOD = 3

.0263	0.0000	.0311	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	.0396	.0415
.0421	.0386	.0504	1.6500	1.5391	1.4577
0.0000	1.2645	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0567	1.0275	1.0217
1.0608	1.0677				

PERIOD = 4

.0263	.0292	0.0000	.0325	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	.0402	.0421
.0428	.0394	.0508	1.6500	1.5391	1.4577
1.3476	0.0000	1.2060	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	1.0443	1.0217	1.0178
1.0504	1.0628				

PERIOD = 5

.0263	.0292	.0311	0.0000	.0337	0.0000
0.0000	0.0000	0.0000	0.0000	.0408	.0428
.0434	.0401	.0511	1.6500	1.5391	1.4577
1.3476	1.2645	0.0000	1.1612	0.0000	0.0000
0.0000	0.0000	0.0000	1.0341	1.0178	1.0132
1.0416	1.0581				

PERIOD = 6

.0263	.0292	.0311	.0325	0.0000	.0349
0.0000	0.0000	0.0000	0.0000	.0415	.0434
.0439	.0408	0.0000	1.6500	1.5391	1.4577
1.3476	1.2645	1.2060	0.0000	1.1319	0.0000
0.0000	0.0000	0.0000	1.0275	1.0132	1.0091
1.0341	0.0000				

PERIOD = 7

.0263	.0292	.0311	.0325	.0337	0.0000
.0358	0.0000	0.0000	0.0000	.0422	.0439
.0445	.0415	0.0000	1.6500	1.5391	1.4577
1.3476	1.2645	1.2060	1.1612	0.0000	1.1040
0.0000	0.0000	0.0000	1.0229	1.0091	1.0054
1.0275	0.0000				

PERIOD = 8

.0263	.0292	.0311	.0325	.0337	.0349
0.0000	.0369	0.0000	0.0000	.0428	0.0000
.0450	.0422	0.0000	1.6500	1.5391	1.4577
1.3476	1.2645	1.2060	1.1612	1.1319	0.0000
1.0888	0.0000	0.0000	1.0178	0.0000	1.0020
1.0229	0.0000				

PERIOD = 9

.0263	.0292	.0311	.0325	.0337	.0349
.0358	0.0000	.0377	0.0000	.0434	0.0000
.0455	.0428	0.0000	1.6500	1.5391	1.4577
1.3476	1.2645	1.2060	1.1612	1.1319	1.1040
0.0000	1.0697	0.0000	1.0132	0.0000	.9990
1.0178	0.0000				

THE VARIANCES OF THE A MATRIX COEFFICIENTS

```
PERIOD =  0
 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0000031  .0000023  .0000027
  .0000028  .0000022  .0000059 0.0000000  .0892767 0.0000000
 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000  .0002174  .0000510  .0001386  .0001821
  .0000862  .0009691
```

```
PERIOD =  1
  .0000022 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0000032  .0000024  .0000028
  .0000030  .0000023  .000005817.6166215 0.0000000  .0034059
 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000  .0002343  .0000582  .0001821  .0001999
  .0001092  .0008918
```

```
PERIOD =  2
 0.0000000  .0000016 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0000033  .0000025  .0000030
  .0000031  .0000024  .000005817.6166215  .0892767 0.0000000
  .0006185 0.0000000 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000  .0002362  .0000800  .0001999  .0002174
  .0001349  .0008184
```

```
PERIOD =  3
  .0000022 0.0000000  .0000018 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000 0.0000000  .0000027  .0000031
  .0000032  .0000026  .000005717.6166215  .0892767  .0034059
 0.0000000  .0004226 0.0000000 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0001140  .0002174  .0002343
  .0001483  .0007492
```

```
PERIOD =  4
  .0000022  .0000016 0.0000000  .0000019 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000 0.0000000  .0000028  .0000032
  .0000033  .0000027  .000005717.6166215  .0892767  .0034059
  .0006185 0.0000000  .0002727 0.0000000 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0001573  .0002343  .0002362
  .0001646  .0006845
```



```
PERIOD =  5
  .0000022  .0000016  .0000018 0.0000000  .0000020 0.0000000
 0.0000000 0.0000000 0.0000000 0.0000000  .0000030  .0000033
  .0000034  .0000028  .000005617.6166215  .0892767  .0034059
  .0006185  .0004226 0.0000000  .0002106 0.0000000 0.0000000
 0.0000000 0.0000000 0.0000000  .0001999  .0002362  .0002508
  .0001821  .0006245
```

```
PERIOD =  6
  .0000022  .0000016  .0000018  .0000019 0.0000000  .0000021
 0.0000000 0.0000000 0.0000000 0.0000000  .0000031  .0000034
  .0000036  .0000030 0.000000017.6166215  .0892767  .0034059
  .0006185  .0004226  .0002727 0.0000000  .0001368 0.0000000
 0.0000000 0.0000000 0.0000000  .0002174  .0002509  .0002648
  .0001999 0.0000000
```

```
PERIOD =  7
  .0000022  .0000016  .0000018  .0000019  .0000020 0.0000000
  .0000022 0.0000000 0.0000000 0.0000000  .0000032  .0000036
  .0000037  .0000031 0.000000017.6166215  .0892767  .0034059
  .0006185  .0004226  .0002727  .0002106 0.0000000  .0001506
 0.0000000 0.0000000 0.0000000  .0002207  .0002648  .0002780
  .0002174 0.0000000
```

```
PERIOD =  8
  .0000022  .0000016  .0000018  .0000019  .0000020  .0000021
 0.0000000  .0000023 0.0000000 0.0000000  .0000033 0.0000000
  .0000038  .0000032 0.000000017.6166215  .0892767  .0034059
  .0006185  .0004226  .0002727  .0002106  .0001368 0.0000000
  .0001200 0.0000000 0.0000000  .0002362 0.0000000  .0002904
  .0002207 0.0000000
```

```
PERIOD =  9
  .0000022  .0000016  .0000018  .0000019  .0000020  .0000021
  .0000022 0.0000000  .0000025 0.0000000  .0000034 0.0000000
  .0000039  .0000033 0.000000017.6166215  .0892767  .0034059
  .0006185  .0004226  .0002727  .0002106  .0001368  .0001506
 0.0000000  .0001650 0.0000000  .0002508 0.0000000  .0003021
  .0002362 0.0000000
```

THE B MATRIX

ROW DESIGNATIONS

```
 4  5  6  7  8  9 10 11 12 13 14 15  8  9 10 11
12 13 14 15 22 23 24 25 26 27 28 29 30 31 32 33
26 27 28 29 30 31 32 33 16 16 16 16 16 16 16 16
```

COLUMN DESIGNATIONS

```
 1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16
17 18 19 20  1  2  3  4  5  6  7  8  9 10 11 12
13 14 15 16 17 18 19 20 13 14 15 16 17 18 19 20
```

PERIOD = 0

0.000	0.000	0.000	0.000	0.000	0.000
-1.051	0.000	0.000	-1.070	0.000	0.000
0.000	0.000	-1.062	-1.187	-1.103	-1.083
0.000	-1.143	0.000	0.000	0.000	0.000
0.000	0.000	-26.013	0.000	0.000	-27.771
0.000	0.000	0.000	0.000	-26.282	-35.024
-29.418	-28.106	0.000	-25.132	0.000	0.000
.435	1.130	.692	.505	0.000	1.700

PERIOD = 1

0.000	0.000	0.000	0.000	0.000	0.000
-1.043	0.000	-1.070	-1.060	0.000	0.000
0.000	0.000	0.000	0.000	0.000	-1.074
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-25.289	0.000	-27.771	-26.833
0.000	0.000	0.000	0.000	0.000	0.000
0.000	-27.198	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.572	0.000	0.000

PERIOD = 2

0.000	0.000	0.000	0.000	0.000	0.000
-1.038	0.000	-1.060	-1.051	-1.115	0.000
0.000	0.000	0.000	0.000	0.000	-1.065
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-24.674	0.000	-26.833	-26.013
-31.666	0.000	0.000	0.000	0.000	0.000
0.000	-26.361	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.561	0.000	0.000

PERIOD = 3

0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-1.051	-1.043	-1.097	0.000
0.000	0.000	0.000	0.000	-1.065	0.000
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	-26.013	-25.289
-30.138	0.000	0.000	0.000	0.000	0.000
-26.361	0.000	0.000	0.000	0.000	0.000
.547	0.000	.561	0.000	0.000	0.000

PERIOD = 4

-1.324	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-1.043	-1.038	-1.082	0.000
0.000	0.000	0.000	0.000	0.000	0.000
-1.098	0.000	-49.090	0.000	0.000	0.000
0.000	0.000	0.000	0.000	-25.289	-24.674
-28.859	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-29.270	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.597	0.000

PERIOD = 5

0.000	0.000	0.000	0.000	0.000	0.000
0.000	-1.060	-1.038	-1.031	-1.070	0.000
0.000	0.000	0.000	0.000	0.000	0.000
-1.085	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	-26.833	-24.674	-24.096
-27.771	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-28.158	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.584	0.000

PERIOD = 6

-1.324	0.000	-1.202	0.000	0.000	0.000
0.000	-1.051	-1.031	-1.026	-1.060	0.000
0.000	0.000	0.000	0.000	0.000	-1.039
0.000	0.000	-49.090	0.000	-38.980	0.000
0.000	0.000	0.000	-26.013	-24.096	-23.573
-26.833	0.000	0.000	0.000	0.000	0.000
0.000	-23.876	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.539	0.000	1.273

PERIOD = 7

-1.324	0.000	0.000	0.000	0.000	0.000
0.000	-1.044	-1.026	-1.021	-1.051	0.000
0.000	0.000	0.000	-1.058	0.000	-1.034
0.000	0.000	-49.090	0.000	0.000	0.000
0.000	0.000	0.000	-25.319	-23.573	-23.098
-26.013	0.000	0.000	0.000	0.000	-25.652
0.000	-23.391	0.000	0.000	0.000	0.000
0.000	.552	0.000	.532	0.000	0.000

PERIOD = 8

-1.324	0.000	-1.202	0.000	-1.137	0.000
0.000	-1.038	0.000	-1.017	-1.044	0.000
-1.154	0.000	0.000	0.000	0.000	0.000
0.000	0.000	-49.090	0.000	-38.980	0.000
-33.561	0.000	0.000	-24.674	0.000	-22.663
-25.319	0.000	-34.046	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.606	0.000
0.000	0.000	.532	0.000	0.000	0.000

PERIOD = 9

-1.324	0.000	-1.202	0.000	0.000	0.000
0.000	-1.031	0.000	-1.013	-1.038	0.000
0.000	-1.128	0.000	0.000	0.000	0.000
0.000	0.000	-49.090	0.000	-38.980	0.000
0.000	0.000	0.000	-24.096	0.000	-22.264
-24.674	0.000	0.000	-32.052	0.000	0.000
0.000	0.000	0.000	0.000	0.000	.643
0.000	0.000	0.000	0.000	0.000	0.000

THE VARIANCES OF THE B MATRIX COEFFICIENTS

PERIOD = 0

0.000	0.000	0.000	0.000	0.000	0.000
.105	0.000	0.000	.172	0.000	0.000
0.000	0.000	.032	.098	.053	.038
0.000	.139	0.000	0.000	0.000	0.000
0.000	0.000	63.078	0.000	0.000	114.536
0.000	0.000	0.000	0.000	18.473	83.234
36.138	24.352	0.000	65.721	0.000	0.000
.005	.087	.020	.008	0.000	.298

PERIOD = 1

0.000	0.000	0.000	0.000	0.000	0.000
.633	0.000	3.286	.564	0.000	0.000
0.000	0.000	0.000	0.000	0.000	.042
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	370.450	0.000	2207.589	359.499
0.000	0.000	0.000	0.000	0.000	0.000
0.000	25.708	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.011	0.000	0.000

PERIOD = 2

0.000	0.000	0.000	0.000	0.000	0.000
.466	0.000	.564	.597	1.073	0.000
0.000	0.000	0.000	0.000	0.000	.041
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	262.145	0.000	359.498	364.137
863.213	0.000	0.000	0.000	0.000	0.000
0.000	23.730	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.011	0.000	0.000

PERIOD = 3

0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	.597	.633	.475	0.000
0.000	0.000	0.000	0.000	.041	0.000
0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	0.000	364.135	370.237
356.683	0.000	0.000	0.000	0.000	0.000
23.730	0.000	0.000	0.000	0.000	0.000
.010	0.000	.011	0.000	0.000	0.000

PERIOD = 4

1.387	0.000	0.000	0.000	0.000	0.000
0.000	0.000	.633	.466	.503	0.000
0.000	0.000	0.000	0.000	0.000	0.000
.046	0.000	1903.339	0.000	0.000	0.000
0.000	0.000	0.000	0.000	370.233	262.073
355.426	0.000	0.000	0.000	0.000	0.000
0.000	0.000	30.983	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.013	0.000

PERIOD = 5

0.000	0.000	0.000	0.000	0.000	0.000
0.000	2.794	.466	.719	.532	0.000
0.000	0.000	0.000	0.000	0.000	0.000
.044	0.000	0.000	0.000	0.000	0.000
0.000	0.000	0.000	1786.526	262.076	390.613
356.504	0.000	0.000	0.000	0.000	0.000
0.000	0.000	28.083	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.012	0.000

PERIOD = 6

1.387	0.000	.348	0.000	0.000	0.000
0.000	.597	.719	.761	.564	0.000
0.000	0.000	0.000	0.000	0.000	.037
0.000	0.000	1903.337	0.000	365.618	0.000
0.000	0.000	0.000	364.134	390.618	400.055
359.498	0.000	0.000	0.000	0.000	0.000
0.000	18.763	0.000	0.000	0.000	0.000
0.000	0.000	0.000	.010	0.000	.124

PERIOD = 7

1.387	0.000	0.000	0.000	0.000	0.000
0.000	.452	.761	.806	.597	0.000
0.000	0.000	0.000	.040	0.000	.037
0.000	0.000	1903.337	0.000	0.000	0.000
0.000	0.000	0.000	264.085	400.056	410.674
364.134	0.000	0.000	0.000	0.000	22.100
0.000	17.785	0.000	0.000	0.000	0.000
0.000	.010	0.000	.009	0.000	0.000

PERIOD = 8

1.387	0.000	.348	0.000	.240	0.000
0.000	.679	0.000	.854	.452	0.000
.049	0.000	0.000	0.000	0.000	0.000
0.000	0.000	1903.346	0.000	365.617	0.000
209.033	0.000	0.000	382.380	0.000	422.522
264.083	0.000	42.367	0.000	0.000	0.000
0.000	0.000	0.000	0.000	.013	0.000
0.000	0.000	.009	0.000	0.000	0.000

PERIOD = 9

1.387	0.000	.348	0.000	0.000	0.000
0.000	.719	0.000	.905	.679	0.000
0.000	.049	0.000	0.000	0.000	0.000
0.000	0.000	1903.338	0.000	365.617	0.000
0.000	0.000	0.000	390.613	0.000	435.586
382.375	0.000	0.000	39.760	0.000	0.000
0.000	0.000	0.000	0.000	0.000	.016
0.000	0.000	0.000	0.000	0.000	0.000

2 THE COVARIANCE MATRICES

THE INITIAL COVARIANCE OF THE STATE VECTOR
AND THE COVARIANCES OF THE EQUATIONS OF MOTION ERROR TERMS

THE INITIAL COVARIANCES OF THE STATE VECTOR

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.32	185.89	1612.11
6473.74	1215.99	167.85	0.00	193.35	0.00
0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	77.96	6741.67	73321.84
370037.24	58863.98	2169.30			

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	3.08	218.73	2668.82
13879.31	1815.42	97.34			

PERIOD = 0

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	116.95	1353.53
5065.39	1390.74	77.71	147.49	0.00	773.40
0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.17	1289.83	9721.82
33530.35	12430.13	1378.76			

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.01	43.71	364.64
1292.18	403.28	62.73			

PERIOD = 1

49.34	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	133.91	1468.66
2226.66	1612.93	81.51	232.80	589.95	0.00
1457.67	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.17	1289.83	9721.82
14465.50	12430.13	1378.76			

26.32	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.01	45.40	374.66
571.37	421.21	63.56			

PERIOD = 2

0.00	78.92	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	151.01	1496.47
541.41	1756.95	85.38	222.83	931.19	1327.38
0.00	1457.67	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.17	1289.83	9721.82
3456.52	12430.13	1378.76			

0.00	31.60	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.01	46.95	384.00
139.64	437.51	64.35			

PERIOD = 3

84.68	0.00	112.59	0.00	0.00	0.00
0.00	0.00	0.00	0.00	168.05	310.76
550.28	1799.93	89.32	174.91	891.32	2095.18
2501.79	0.00	1457.67	0.00	0.00	0.00
0.00	0.00	0.00	0.00	1289.83	1983.98
3456.52	12430.13	1378.76			

45.17	0.00	35.81	0.00	0.00	0.00
0.00	0.00	0.00	0.00	48.38	80.15
142.57	452.44	65.12			

PERIOD = 4

133.67	135.45	0.00	137.58	0.00	0.00
0.00	0.00	0.00	0.00	184.83	315.85
546.25	1028.69	93.33	46.66	699.62	2005.47
3948.91	2501.79	0.00	1457.67	0.00	0.00
0.00	0.00	0.00	0.00	1289.83	1983.98
3456.52	6948.97	1378.76			

71.29	54.23	0.00	39.32	0.00	0.00
0.00	0.00	0.00	0.00	49.71	81.83
145.34	260.64	65.85			

PERIOD = 5

127.95	213.80	193.23	0.00	173.61	0.00
0.00	0.00	0.00	0.00	198.54	313.54
554.13	931.71	97.40	1.37	186.63	1574.15
3779.84	3948.91	2501.79	0.00	1457.67	0.00
0.00	0.00	0.00	0.00	1289.83	1983.98
3456.52	6167.44	1378.76			

68.24	85.59	61.46	0.00	42.33	0.00
0.00	0.00	0.00	0.00	50.95	83.42
147.96	237.68	66.56			

PERIOD = 6

100.43	204.65	305.01	236.13	0.00	175.49
0.00	0.00	0.00	0.00	202.03	318.06
523.74	949.35	0.00	90.36	5.50	419.91
2966.91	3779.84	3948.91	2501.79	0.00	1457.67
0.00	0.00	0.00	0.00	1289.83	1983.98
3223.11	6167.44	0.00			

53.56	81.93	97.01	67.49	0.00	44.96
0.00	0.00	0.00	0.00	52.11	84.93
140.30	243.61	0.00			

PERIOD = 7

26.79	160.64	291.95	372.71	297.96	0.00
207.48	0.00	0.00	0.00	78.74	322.39
163.67	966.03	0.00	88.10	361.46	12.37
791.30	2966.91	3779.84	3948.91	2501.79	0.00
1457.67	0.00	0.00	0.00	505.89	1983.98
994.45	6167.44	0.00			

14.29	64.31	92.86	106.52	72.64	0.00
47.29	0.00	0.00	0.00	20.87	86.36
43.97	249.16	0.00			

PERIOD = 8

.79	42.84	229.16	356.76	470.32	301.20
0.00	58.40	0.00	0.00	79.95	0.00
165.66	959.92	0.00	225.77	352.42	813.28
23.30	791.30	2966.91	3779.84	3948.91	2501.79
0.00	434.17	0.00	0.00	505.89	0.00
994.45	6167.44	0.00			

.42	17.15	72.89	101.96	114.67	77.16
0.00	14.71	0.00	0.00	21.27	0.00
44.62	254.39	0.00			

PERIOD = 9

51.89	1.26	61.12	280.03	450.18	475.42
356.10	0.00	.23	0.00	81.10	0.00
167.58	974.66	0.00	22.70	903.06	792.94
1532.83	23.30	791.30	2966.91	3779.84	3948.91
2501.79	0.00	1.48	0.00	505.89	0.00
994.45	6167.44	0.00			

27.67	.51	19.44	80.03	109.76	121.79
81.17	0.00	.05	0.00	21.66	0.00
45.24	259.33	0.00			

THE COVARIANCES OF THE ERROR TERM OF THE OBSERVER

PERIOD = 1

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	120.82	1435.36
5403.63	1456.63	76.96	9.22	0.00	907.67
0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.29	4656.09	67886.29
261274.85	75024.55	1015.57			

0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.05	157.78	2546.27
10068.93	2434.08	46.20			

PERIOD = 2

49.71	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	139.34	1566.73
2377.79	1705.00	80.90	14.55	331.84	0.00
2599.48	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.28	5666.77	75754.26
111332.03	92608.77	1090.07			

46.93	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.05	199.45	2919.39
4397.48	3138.15	50.25			

PERIOD = 3

0.00	80.41	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.03	158.24	1598.04
578.72	1867.76	84.92	13.93	523.79	1557.83
0.00	4622.60	0.00	0.00	0.00	0.00
0.00	0.00	0.00	1.21	6746.81	74822.86
26315.99	101869.94	1168.52			

0.00	100.20	0.00	0.00	0.00	0.00
0.00	0.00	0.00	.05	245.58	2955.41
1063.16	3585.53	54.54			

PERIOD = 4

85.32	0.00	116.14	0.00	0.00	0.00
0.00	0.00	0.00	0.00	177.31	332.17
588.75	1915.80	89.03	10.93	501.37	2458.92
4461.46	0.00	7335.94	0.00	0.00	0.00
0.00	0.00	0.00	0.00	7876.30	15104.86
26065.26	99893.69	1250.96			

80.55	0.00	180.23	0.00	0.00	0.00
0.00	0.00	0.00	0.00	295.42	610.23
1075.13	3636.01	59.08			

PERIOD = 5

134.68	138.01	0.00	143.13	0.00	0.00
0.00	0.00	0.00	0.00	196.34	337.93
583.79	1096.18	93.23	2.92	393.54	2353.64
7042.10	7933.73	0.00	9088.14	0.00	0.00
0.00	0.00	0.00	0.00	9033.89	14960.95
24690.31	54925.07	1337.40			

127.14	171.97	0.00	245.15	0.00	0.00
0.00	0.00	0.00	0.00	348.15	617.11
1038.19	2060.12	63.88			

PERIOD = 6

128.91	217.83	199.32	0.00	182.92	0.00
0.00	0.00	0.00	0.00	212.02	335.08
592.70	993.92	97.51	.09	104.98	1847.44
6740.59	12522.86	12590.62	0.00	12504.52	0.00
0.00	0.00	0.00	0.00	9927.02	14171.77
24503.50	48057.88	1427.78			

121.69	271.44	309.32	0.00	363.10	0.00
0.00	0.00	0.00	0.00	392.11	595.90
1048.93	1852.04	68.93			

PERIOD = 7

101.19	208.51	314.62	245.66	0.00	184.86
0.00	0.00	0.00	0.00	215.95	340.20
560.63	1013.78	0.00	5.65	3.09	492.82
5290.90	11986.72	19873.48	15598.00	0.00	11307.80
0.00	0.00	0.00	0.00	9820.03	14064.52
22693.06	47467.03	0.00			

95.52	259.82	488.24	420.75	0.00	348.74
0.00	0.00	0.00	0.00	396.73	602.06
987.78	1874.89	0.00			

PERIOD = 8

26.99	163.66	301.15	387.76	313.95	0.00
220.99	0.00	0.00	0.00	84.08	345.09
175.33	1032.60	0.00	5.51	203.32	14.51
1411.12	9408.74	19022.60	24620.38	21461.49	0.00
14296.94	0.00	0.00	0.00	3644.44	13968.66
6958.42	46955.36	0.00			

25.48	203.94	467.34	664.13	623.18	0.00
463.85	0.00	0.00	0.00	150.32	608.03
307.66	1896.99	0.00			

PERIOD = 9

.79	43.65	236.38	371.15	495.55	317.28
0.00	61.90	0.00	0.00	85.44	0.00
177.59	1025.01	0.00	14.11	198.24	954.47
41.56	2509.39	14931.36	23566.24	33875.57	19407.52
0.00	3478.41	0.00	0.00	3613.65	0.00
6919.33	44429.93	0.00			

.75	54.39	366.83	635.70	983.65	598.55
0.00	117.87	0.00	0.00	151.95	0.00
310.47	1832.63	0.00			

PERIOD = 10

52.28	1.29	63.04	291.33	474.33	500.80
379.29	.01	.24	.01	86.75	0.00
179.77	1041.65	0.00	1.42	507.97	930.60
2733.51	73.90	3982.32	18497.84	32425.17	30633.47
24537.84	0.00	14.20	0.00	3586.32	0.00
6883.81	44054.70	0.00			

49.35	1.60	97.84	498.98	941.53	944.76
796.10	0.00	.50	0.00	153.52	0.00
313.19	1852.44	0.00			

3 THE TARGET VALUES FOR THE STATE AND CONTROL VARIABLES

THE STATE VARIABLES

PERIOD = 0

0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	6.90	51.88	193.92	450.90	146.43	40.17	

PERIOD = 1

0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	.91	45.21	192.60	388.67	174.50	27.92	

PERIOD = 2

17.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	.92	51.95	209.29	260.11	203.02	29.36	

PERIOD = 3

0.00	28.37	0.00	0.00	0.00	0.00	0.00	0.00
0.00	.91	58.76	213.23	129.37	221.45	30.83	

PERIOD = 4

22.67	0.00	40.98	0.00	0.00	0.00	0.00	0.00
0.00	0.00	65.54	98.02	131.49	226.86	32.33	

PERIOD = 5

28.48	37.17	0.00	50.33	0.00	0.00	0.00	0.00
0.00	0.00	72.22	99.62	130.44	173.40	33.86	

PERIOD = 6

27.87	46.69	53.69	0.00	63.84	0.00	0.00	0.00
0.00	0.00	77.67	98.82	132.32	166.69	35.41	

PERIOD = 7

24.69	45.68	67.45	65.93	0.00	64.49	0.00	0.00
0.00	0.00	79.03	100.24	129.50	169.84	0.00	

PERIOD = 8

12.75	40.47	65.99	82.83	83.63	0.00	76.48	0.00
0.00	0.00	49.15	101.60	72.85	172.81	0.00	

PERIOD = 9

2.19	20.90	58.46	81.04	105.07	84.48	0.00	39.37
0.00	0.00	49.90	0.00	73.73	171.61	0.00	

PERIOD = 10

17.75	3.59	30.19	71.80	102.80	106.14	100.20	0.00
2.62	0.00	50.62	0.00	74.58	174.24	0.00	

THE CONTROL VARIABLES

PERIOD = 0

0.00	0.00	0.00	0.00	0.00	0.00	.31
0.00	0.00	86.77	0.00	0.00	0.00	0.00
5.75	13.28	20.95	10.89	0.00	13.66	

PERIOD = 1

0.00	0.00	0.00	0.00	0.00	0.00	.05
0.00	2.40	16.81	0.00	0.00	0.00	0.00
0.00	0.00	0.00	133.38	0.00	0.00	

PERIOD = 3

0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	5.78	7.08	19.83	0.00	0.00	0.00
.91	0.00	116.91	0.00	0.00	0.00	

PERIOD = 2

0.00	0.00	0.00	0.00	0.00	0.00	.07
0.00	13.78	7.62	9.02	0.00	0.00	0.00
0.00	0.00	0.00	132.96	0.00	0.00	

PERIOD = 4

3.37	0.00	0.00	0.00	0.00	0.00	0.00
0.00	5.37	9.54	13.64	0.00	0.00	0.00
0.00	0.00	0.00	0.00	57.24	0.00	

PERIOD = 5

0.00	0.00	0.00	0.00	0.00	0.00	0.00
1.01	7.23	6.09	11.86	0.00	0.00	0.00
0.00	0.00	0.00	0.00	10.04	0.00	

PERIOD = 6

4.41	0.00	11.11	0.00	0.00	0.00	0.00
4.66	4.62	5.50	10.98	0.00	0.00	0.00
0.00	0.00	0.00	4.55	0.00	35.41	

PERIOD = 7

5.54	0.00	0.00	0.00	0.00	0.00	0.00
3.81	4.31	2.86	10.18	0.00	0.00	0.00
0.00	29.54	0.00	57.56	0.00	0.00	

PERIOD = 8

5.42	0.00	14.55	0.00	7.87	0.00	0.00
2.50	0.00	2.67	13.31	0.00	34.74	0.00
0.00	0.00	101.60	0.00	0.00	0.00	

PERIOD = 9

4.81	0.00	18.28	0.00	0.00	0.00	0.00
2.33	0.00	2.50	8.72	0.00	0.00	37.07
0.00	0.00	0.00	0.00	0.00	0.00	

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