

Optimal Decision in the Broiler
Producing Firm:
A Problem of Growing Inventory

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Optimal producer behavior is examined in the production of products which may be regarded as growing inventories. Any one of several types of livestock produced by farm firms provides an example of such a product. This study focuses on the broiler producing firm.

Weight-feed relations, derived from underlying weight and feed "growth functions," are regarded as deterministic. Broiler firms are assumed to be confronted with probabilistic prices with known probability distribution. Optimal policy takes the form of a set of cutoff prices, a cutoff price for each marketing age of the broiler over a relevant price range. If price at a given age (week) is above the cutoff price, the producer sells; if below, he keeps his flock for at least one more week.

Most of the results are derived for the homogeneous case (that is, the probability distribution of prices is assumed to be the same in each week), but this assumption is subsequently relaxed to derive one form of an interseasonal decision model. Optimal policies are derived under two alternative assumptions regarding the form of the probability distribution of prices—the normal and the uniform. The sensitivity of the homogeneous model to changing variance of prices is also examined.

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OPTIMAL DECISION IN THE BROILER PRODUCING FIRM: A PROBLEM OF GROWING INVENTORY¹

INTRODUCTION

PRODUCTION DECISION models involving growing inventories have long been discussed by economists. The problem of time and timing has roots in early economic thinking, and kernels of it can be found in the discussion between D. Ricardo (1895) and one of his disciples, J. R. McCulloch. McCulloch claimed that time by itself does not bear any "fruits." As an example, he considers two barrels of wine: the first contains "unfinished" wine, though the treading process is finished, and the second "finished" wine which will not be improved by additional time. After one year, the first barrel will have increased its value, but the second will have not, though time "operated" equally on both barrels! His conclusion: time itself cannot produce any effect. He strengthens this argument by another example from plantations. In growing trees, even though all factors of production were implemented, the production is not finished—additional time is needed. And he explains this phenomenon in an illustrative way. There is a "machine" in the tree that needs time to operate. Applying this to the wine example, the unfinished wine is like a machine for producing wine. Ricardo refutes McCulloch's argument by raising the question: How come the same machine in the tree produces different results under different interest rates?

In a sense, several facets of the problems discussed in the present paper appear in a heuristic form within this early debate. It is the process of "growing inventory" that serves as the focus of this discussion. But whereas McCulloch was indicating the growth process over time, whether through quantitative changes in volume or weight (the growing tree) or through qualitative changes (the vintage of wine), Ricardo (1895) was concerned with the opportunity costs of holding inventory over time, manifesting themselves through the interest rate.

Growth relations in broiler production

One of the most important problems the production economist faces in investigating the production of livestock is how to incorporate the complexity of a biological process into the concept of a production function. The classification concept of a production function means a transformation of a set of inputs controlled by the producer into a given output. Heady and Dillon (1961, pp. 323-30) suggest introducing the growth process through selection of the appropriate algebraic form to express the input-output transformation.

The main criticism of this approach is that it confuses concepts of growth

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and production functions. The typical growth relationships, such as the sigmoid curve, are relations between weight of bird and age. The assumption that the production function has the same shape as the growth curve calls for some theoretical justification. Because both weight and feed consumption bear a high correlation with age of bird, the correlation between weight and feed will be strong.

The problem is presented by Baum and Walkup (1953) and Brown and Arscott (1958). The approach most closely related to that adopted here is that of Hoepner and Freund (1964) who suggest a model constructed of two parts:

Static:

$W = b + b_1 F + b_2 F^2$ where W refers to body weight in grams and F refers to total feed consumed.

Nonstatic:

$$W = \frac{M}{1 + Ae^{-\gamma T}}$$

$$F = d_0 + d_1 T + d_2 T^2.$$

The reasoning behind this model is that both weight and feed are functions of age. This idea has been used in our own formulations here.

Decision rules

The conventional framework of a decision model for broiler production adopts as the objective maximization of profit per year considering one limiting resource—floor capacity. This amounts to maximizing average profit per unit of capacity.

For formulations within this framework, see Brown and Arscott (1958) and Hoepner and Freund (1964). Faris (1960)

suggests a simple formula for an optimal replacement policy in the case of a short production period with revenue being realized by the sale of the asset:

“... the present lot should be carried only to the point where marginal net revenue from it equals maximum average net revenue anticipated from the subsequent lot. To carry the present lot beyond this point would yield additions to net revenue less than the maximum average anticipated in the future.”

The Faris results link the simple theory of replacement with the dynamic approach to the problem.

Sequential stochastic decision models have not been applied directly to problems here under consideration. However, attention has been given to the concepts of dynamic programming and Markov processes in the theory of inventory and waiting lines in general, of which the replacement problem is a special simple case.

A general mathematical framework accommodating most types of replacement decisions is provided by the dynamic programming model. In the discrete stochastic case, Howard (1960) has suggested a solution for the replacement problem.

The Howardian model is based on the integration of (1) transition probability matrices defining complete Markov chains, which are determined exogenously by forces not controlled by the producer—for example, the matrix of chance failure or loss—and (2) alternative return functions attached to each of the exogenous transition probabilities.

Burt (1965) analyzes the problem of optimal replacement under risk in a special case of the Howard model, for which an analytic solution is derived.

This is a model for economic analysis of asset life under conditions of chance failure or loss. The solution suggested by Burt may be summarized as follows: Define the conditional expected value of net revenue during a time interval for an asset of age t (excluding cost of planned replacement) as

$$R_t = P_t H_t - (1 - P_t) D_t$$

where

P_t = probability that an asset of age t will reach age $t + 1$ with normal productivity

H_t = net revenue associated with an asset of age t in the absence of replacement due to random causes

and

D_t = cost of replacement caused by random factors.

The criterion function is

$$V(T) = \frac{1}{1 - \beta}$$

$$\left[\sum_{t=1}^T W_t R_t - W_T P_T C_T \right] / \sum_{t=1}^T W_t$$

where

W_t = the (discounted) probability that an asset reaches age t

C_t = voluntary replacement cost (cost of a new asset minus terminal value of the used one)

T = planned replacement age

and

$\beta = 1/(1 + i)$, where i is the relevant interest rate for discounting.

This criterion is simply the capitalization of a weighted average of expected net revenues into perpetuity. The optimal policy is one which maximizes $V(T)$ with respect to T . Several straightforward methods of solution can be applied as pointed out by Burt. One of them puts marginal conditions for optimal replacement in the form of two inequalities:

$$V(T) \geq V(T + 1)$$

$$V(T) \geq V(T - 1).$$

Simple elementary operations supply us with the optimal T . The replacement process can be formulated as a finite Markov chain defined for each choice of replacement age T . Given the replacement age T , the transition probability matrix is determined completely by the probability of forced replacement during the period, which is $(1 - P_t)$ for an asset of age t .

Burt's model is restricted to the case where the evolution of the system from one state to another is determined by forces exogenous to the system. The decision made by the producer is with respect only to planned replacement at age (T). This model may be sufficient for a fixed asset, where long-run considerations are dominant. But, in the case of a growing inventory, there are additional problems of relations between stocks and flows; and situations arise where current decisions influence transition of the system from one state to another. Hence, additional modifications, where the transition probabilities turn out to be endogenous in the model, are necessary.

GROWING INVENTORY MODELS

Some problems faced by the broiler producer extend to a broader class of agricultural commodities. The class of "growing inventory" commodities includes livestock raised for marketing, stocks of wine going through a process of quality improvement, and growing timber. However, we shall concentrate on the specific case of the broiler producer, keeping in mind the more general framework of the problem; and narrow most of the examples to the Israeli broiler producer. From studying the Israeli case, meaningful generalizations can be revealed that may be applied to completely different environmental and institutional situations.

In selecting the appropriate models, several essential characteristics of broiler production should be taken into account:

1. The period of production per flock is short.
2. By defining the production function per flock, we abandon the classical approach which considers the production relation for some prespecified unit of time (month, year, etc.).
3. The weight-feed relation is the most important factor in broiler production.
4. The weight of the bird is subject

to a physiological growth pattern over its life period.

5. The feed consumption pattern over the bird's life period is governed by factors like feed maintenance requirements, stomach capacity, and the weight-growth pattern.
6. The producer is given a recommended feed composition, to be fed *ad libitum*.
7. Quality of the carcass can be considered as a function of age. In the term "quality" we include all qualitative factors influencing the preferences of the consumers.
8. The time element should be put in its proper perspective—at the level of an individual flock production function. It is here that time should be introduced into the broader aspects of the continuing production process by the firm.
9. The firm operates under conditions only part of which are under control, while others introduce elements of uncertainty into the decision process.

The last two features are best dealt with in the framework of the stochastic model presented on pages 19–28. In the following section we shall demonstrate how the element of time can be introduced in a simple deterministic case.

A Deterministic Formulation

The growth functions

In the following discussion, we use specific forms of algebraic functions having certain desirable mathematical properties. We regard the empirical results based on these functional forms (see pages 31–35) as "acceptable," but our

empirical examination of alternative functional forms has not been sufficiently exhaustive to support a claim that the specific forms chosen are in some sense "best." Other forms may be more appropriate, but the specific form of function is considered not to be of primary importance here.

Production relations for a simple growth process. — We observe that both weight (W) and feed (F) are functions of age of bird (x). This defines the following two functions:

$$W = f_1(x) \quad (1)$$

$$F = f_2(x). \quad (2)$$

But from an economic point of view, we are interested in the following function:

$$W = g(F) = [f_2^{-1}(F)].$$

There are various specific functional forms which might be adopted to represent these relations. We select the following:

$$W = e^{\alpha_0 - \alpha_1(1/x)} \quad \alpha_1 > 0 \quad (4)$$

$$F = e^{\beta_0 - \beta_1(1/x)} \quad \beta_1 > 0. \quad (5)$$

Equations (4) and (5) imply asymptotic levels of weight and feed consumption as age increases, and they allow for varying marginal growth rates.

Marginal rates of growth and inflection points may be derived

$$\frac{dW}{dx} = \alpha_1 \frac{W}{x^2} > 0; \quad (6)$$

$$\frac{d^2W}{dx^2} = \left(\frac{\alpha_1^2}{x^4} - \frac{2\alpha_1}{x^3} \right) W.$$

The point of inflection is at $x = \alpha_1/2$ corresponding to $d^2W/dx^2 = 0$. Similar results are derivable for equation (5), substituting β_1 and F for α_1 and W , respectively.

Equations (4) and (5) supply important information about the relation between weight and feed and establish a unique functional relation. From equations (4) and (5) we derive:

$$W = AF^\delta \quad (7)$$

where $\delta = \alpha_1/\beta_1$ and $A = \text{antilog}(\alpha_0 - \delta\beta_0)$.

Though this relation has the familiar Cobb-Douglas production function form, this is not the classical production function but a derived relation between weight and feed. Time enters through the growth curves and not through fixing the unit of time on which we observe production.

The above describes a simple weight-feed relation for a given flock. To allow for different growth rates for different flocks, we may write:

$$W_i = e^{\alpha_0 - \alpha_{1i}(1/x)} \quad (8)$$

$$F_i = e^{\beta_0 - \beta_{1i}(1/x)} \quad (9)$$

$$W_i = A_i F_i^{\delta_i} \quad (10)$$

where

$i = \text{an index for flock}$

$\delta_i = \alpha_{1i}/\beta_{1i}$

and

$A_i = \text{antilog}(\alpha_0 - \delta_i\beta_0)$.

We assume that the asymptotic levels for F and W are the same for all flocks, but the growth coefficients α_1 and β_1 may vary over flocks. It seems plausible to assume:

$$\alpha_{1i} = f(Y_{1i}, Y_{2i}, \dots, Y_{Hi})$$

and

$$\beta_{1i} = f(Z_{1i}, Z_{2i}, \dots, Z_{Ti})$$

where the variables Y and Z refer to factors such as breed, quality of feed,

management, seasonality, etc.²

But the above represents results of a special case. Because the quantity of feed input is not controlled (the bird determines the quantity consumed *ad libitum*), a one-to-one correspondence exists between feed and weight. And, as a result, time vanishes from the production relations in equations (7) and (10). The picture changes as we consider additional facets through which time manifests itself.

Maintenance feed requirements.

—Maintenance feed requirements represent an element of cost which does not contribute directly to an increase in production. But maintenance feed is plausibly regarded as a function of age of bird. Therefore, feed used for maintenance should be introduced into the production relation in a time-consuming production process. Accordingly, the simple weight-feed relation is replaced by new ones, with two dimensions to the role of time: (1) through the growth process and (2) through the feed maintenance requirements.

Introducing the relation Cx^γ to represent feed maintenance requirements, equation (5) is replaced by:

$$F = Cx^\gamma e^{-\beta_1(1/x)} \quad (5.1)$$

and, allowing for flock effect in the growth coefficient, equation (9) becomes

$$F_i = Cx^\gamma e^{-\beta_{1i}(1/x)} \quad (9.1)$$

where $0 \leq \gamma < 1$ is the maintenance coefficient and $0 \leq \beta_1, \beta_{1i} < \infty$ are feed growth coefficients.³ Equations (4) and (8) remain unchanged.

It is important to distinguish between maintenance and growth coefficients because growth feed follows the usual growth cycles, while maintenance feed can be assumed to have a constant elasticity. As discussed later, one can compare the growth coefficient β_1 with the growth coefficient α_1 and might hypothesize equality of the two coefficients.

Investigating the mathematical properties of equation (5.1) helps in examining decision rules. The first derivative of equation (5.1) is:

$$\frac{dF}{dx} = \frac{F}{x} \left(\frac{\beta_1}{x} + \gamma \right) \quad (11)$$

Under the assumptions of the revised feed equation, there is no maximum or asymptotic level because both $\beta_1 > 0$ and $\gamma > 0$.

To obtain more information about

² The assumption that the asymptotic levels for F and W are the same for all flocks is open to question because certain factors (for example, breed) specified as affecting the growth coefficients α_1 and β_1 might reasonably be expected to affect also the asymptotic levels. Given suitable data, the hypothesis of uniform asymptotic levels over flocks could be tested; and, if rejected, the analysis could be easily extended to accommodate asymptotic levels varying over flocks.

Data available for the present study were not adequate for estimating relations allowing for varying asymptotic levels nor were data adequate to permit inclusion of separate Y and Z factors in estimating the growth coefficients. Therefore, a single dummy variable is introduced to represent "flock effect" in each relation. Then, defining

$$\begin{aligned} \alpha_{1i} &= a_0 + a_1 Y_{1i} \\ \beta_{1i} &= b_0 + b_1 Z_{1i} \end{aligned}$$

our growth relations become

$$\begin{aligned} W_i &= e^{\alpha_0 - \alpha_0(1/x) - \alpha_1 Y_{1i}(1/x)} \\ F_i &= e^{\beta_0 - b_0(1/x) - b_1 Z_{1i}(1/x)} \end{aligned}$$

Empirical results based on this formulation are summarized in table 2, page 33.

³ The boundaries $0 \leq \gamma < 1$ assume that the maintenance feed consumption increases at a decreasing rate (based on Brody, 1964).

the feed relation, consider the second derivative:

$$\frac{d^2F}{dx^2} = \frac{F}{x^2} \left[\frac{\beta_1^2}{x^2} - \frac{2\beta_1(1-\gamma)}{x} - \gamma(1-\gamma) \right]. \tag{12}$$

Setting this expression equal to zero and cancelling and multiplying by $-x^2$, we obtain:

$$\gamma(1-\gamma)x^2 + 2\beta_1(1-\gamma)x - \beta_1^2 = 0. \tag{13}$$

We can solve equation (13) for the inflection point x^* :

$$x^* = \frac{-\beta_1(1-\gamma) + \beta_1\sqrt{1-\gamma}}{\gamma(1-\gamma)}. \tag{14}$$

The negative sign in front of the root is omitted because we exclude the possibility of negative values of age (x). With rearrangement of terms and simple algebraic manipulation, equation (14) is brought to the following simple form:

$$x^* = \frac{\beta_1}{\gamma} \left(\frac{1}{\sqrt{1-\gamma}} - 1 \right) = \beta_1 \phi(\gamma). \tag{15}$$

It is of interest to compare equation (15) with the inflection point obtained from equation (5). Evaluate $\phi(\gamma)$ in equation (15) as γ approaches zero. This can be done through the application of L' Hopital's rule to $\phi(\gamma)$:

$$\frac{\left(\frac{1}{\sqrt{1-\gamma}} - 1 \right)'}{\gamma'} \tag{16}$$

$$= \frac{1}{2} (1-\gamma)^{-3/2} \rightarrow \frac{1}{2}, \text{ for } \gamma \rightarrow 0.$$

For $\gamma = 0$, equation (5.1) becomes simply equation (5).

Considerable attention has been given to possible achievements in the field of broiler breeding. Hence, it is of interest to investigate the influence of a change in the coefficients of the feed equation on the inflection point x^* . Through controlling the inflection point, the breeder can influence the profitability of broiler production in general and at given ages in particular.

The inflection point is determined by two factors— β_1 and $\phi(\gamma)$. Consider first, for given β_1 the behavior of $\phi(\gamma)$ as γ increases between 0 and 1. This can be determined by considering the sign of the derivative $d\phi(\gamma)/d\gamma$. To simplify the

expression $\phi(\gamma) = \frac{1}{\gamma} \left(\frac{1}{\sqrt{1-\gamma}} - 1 \right)$, let us use the following binomial expansion for $\frac{1}{\sqrt{1-\gamma}}$:

$$(1-\gamma)^{-1/2} = 1 + \frac{1}{2}\gamma + \frac{3}{(4)(2!)}\gamma^2 + \dots + \frac{(3)(5)\dots(2n-1)}{(2)^n(n!)}\gamma^n + \dots \quad 0 < \gamma < 1.$$

Substituting into $\phi(\gamma)$ we obtain:

$$\phi(\gamma) = \frac{1}{2} + \frac{3}{(2)^2(2!)}\gamma + \dots + \frac{(3)(5)\dots(2n-1)}{(2)^n(n!)}\gamma^{n-1} + \dots$$

In our case $0 \leq \gamma < 1$. Therefore, it is obvious that $1/2$ is a lower bound for $\phi(\gamma)$. From this expansion, it can be verified that $d\phi(\gamma)/d\gamma > 0$ because all coefficients of γ are positive. The empirical meaning of this result is that an increase of the maintenance feed coefficient (γ) through the range 0 to 1 will shift the inflection point x^* to the right.

For given values of the maintenance feed coefficient, the inflection point is a linear function of β_1 :

$$x^* = \phi(\gamma^0) \beta_1 = k\beta_1.$$

The production relations can be derived, as in equation (10), from equations (4) and (5.1) to yield:

$$W = AF^{\delta_{1i}}x^{\delta_{2i}} \quad (17.1)$$

$$W_i = A_i F_i^{\delta_{1i}} x^{\delta_{2i}} \quad (17.2)$$

where

$$i = 1, 2, \dots, N \text{ flocks}$$

$$A_i = \text{antilog} (\alpha_0 - (\alpha_{1i}/\beta_{1i}) \log C)$$

and

$$\delta_{1i} = \alpha_{1i}/\beta_{1i} > 0;$$

$$\delta_{2i} = -(\alpha_{1i}/\beta_{1i}) \gamma < 0.$$

Age of flock appears explicitly in this case as a result of the maintenance requirements. The coefficient δ_{2i} is negative and can be explained intuitively in the following way: For a given feed quota, holding the flock for a longer time requires drawing on the reserves of accumulated "fatness" in order to maintain the bird.

It may be assumed that the "net" growth coefficients of weight and feed are equal, that is, $\alpha_1 = \beta_1$. Note that

$$\frac{\partial W}{\partial x} = \alpha_1 \frac{W}{x^2} \Rightarrow \alpha_1 = \frac{\partial W}{\partial x} \frac{x^2}{W} = \mu_{W/x} x$$

$$\frac{\partial \bar{F}}{\partial x} = \beta_1 \frac{\bar{F}}{x^2} \Rightarrow \beta_1 = \frac{\partial \bar{F}}{\partial x} \frac{x^2}{\bar{F}} = \mu_{\bar{F}/x} x$$

where $\bar{F} = F/x^\gamma$. Both α_1 and β_1 are "rates" of growth referring to the same time units, do not depend on the units of weight or feed, and describe closely related growth processes. Hence, it is reasonable to assume that these coefficients are equal.

Equality of the two growth coefficients justifies the following expression:

$$\frac{F}{W} = B_0 x^\gamma \quad (18)$$

where equation (18) is derived from equation (17) with $\delta_{1i} = 1$ and $B_0 = 1/A_0$. Equation (18) focuses attention on pertinent information needed by the producer, namely, the feed-weight conversion ratio. It is of major importance in evaluating the profitability of production because of the relative importance of feed in the cost of production.

As pointed out on page 6, some explanation of the reasoning behind the distinction between the growth coefficient and the maintenance-feed coefficient is called for. We interpret the growth coefficient as the rate of net growth of weight or feed consumption in relation to age of bird. Because both of the growth coefficients, α_1 and β_1 , describe the same growth process—one of them through the weight function and the other through the feed function—and because both are expressed in terms of rates, we postulated equality between the two. We interpret the maintenance feed coefficient as measuring the elasticity of feed consumption with respect to age, given the weight of the bird—that is, the feed consumption at each age needed for maintaining the bird at a given weight. We do not pretend to investigate the nature of the physiological process; but this conceptual differentiation between the two processes is relevant to the economic decision process.

The problem is how to incorporate in one framework of analysis both the intra-temporal and the intertemporal relations of weight and feed with respect to age of bird. In this example, we do it through assuming that the growth rate represents an "instantaneous" conversion of feed into increments of weight, and, hence, this represents an "intra-temporal" relation. On the other hand,

TABLE 1
EXPENDITURES FOR CHICKEN MEAT
BY URBAN FAMILIES
SOUTHERN UNITED STATES, 1954

Income	Expenditure per week	Quantity purchased	Price
	(V)	(q)	(P)
dollars	dollars	pounds	cents per pound
Under 2,000.....	1.11	2.22	50.0
2,000-2,999.....	1.42	2.76	51.4
3,000-3,999.....	1.39	2.59	53.7
4,000-4,999.....	1.05	2.05	51.2
5,000-5,999.....	1.29	2.54	50.8
6,000-7,999.....	1.31	2.42	54.1
8,000-9,999.....	1.51	2.68	56.3
10,000 and above...	1.99	3.35	59.4

Source: U. S. Department of Agriculture, "Household Food Consumption Survey, 1955," *Food Consumption of Households in the South*, Report 4, Table 10, p. 29.

the maintenance coefficient represents the cost, measured in feed, involved in the "intertemporal" relation. In the special case where only growth coefficients are specified (as above), age vanishes from the weight-feed relation.

The quality function.—The concept of quality has been introduced into the theory of consumer demand and some aspects of it into the theory of the firm in the last two decades (Dorfman and Steiner, 1954). First attention was drawn in empirical research to the importance of quality variations in consumer demand, especially in relation to income (Prais and Houthakker, 1955).

Let $V_i = P_i q_i$ be family expenditure on commodity i and M be family income. Then

$$M_{V_i/M} = \frac{\delta V_i}{\delta M} \frac{M}{V_i} = \frac{\delta q_i}{\delta M} \frac{M}{q_i} + \frac{\delta P_i}{\delta M} \frac{M}{P_i} \quad (19)$$

$$= M_{q_i/M} + M_{P_i/M}.$$

Here, the elasticity of expenditure on commodity i is decomposed into two parts: The first term ($M_{q_i/M}$) is the elas-

ticity of quantity of commodity i with respect to income; and the second term ($M_{P_i/M}$) is the elasticity of quality with respect to income.

Table 1 summarizes some relevant data on expenditure for chicken meat by urban families in the southern United States. The upward drift in prices (column 4) may be attributable to differences in quality of meat bought by higher income families. As Houthakker (1952-53) points out the classical theory of consumption (as in the writings of Hicks, Samuelson, and others) ignores qualities altogether because varieties, if any, of any item of consumption are treated as different commodities (see Theil, 1952-53).

"Since the consumer appears as a buyer, these quantities cannot be negative; indeed, they must be positive, for the more interesting conclusions from the theory, such as the Slutsky equation, the negativity of the own-price substitution effect and the theorem on group demand (Hicks, 1939, pp. 311 and 312) hold only when quantities may change in either direction, so that they cannot be zero. This implies that a commodity . . . has to be very narrowly specified" (Houthakker, 1952-53, p. 555).

Both Theil and Houthakker have suggested introducing quality changes into the theory of demand through changes in price of the commodity. This creates several conceptual difficulties. Among others, it ignores the supply side. Accordingly, an attempt is made here to recognize quality explicitly in the formulation of production decisions.

In the case of broiler production, we assume that quality is related to age of bird. For our purpose, we use the term quality to include both qualitative factors—such as flavor, taste, color, and

others—and quantitative factors—such as size of bird—that influence the scale of consumers' preferences. This group of factors is introduced as a trend, hypothesizing a rapid rate of quality growth at early ages, followed by a slowly decreasing rate of growth. The quality function is, in fact, an index used to deflate the weight of the bird so that the resulting product is a homogeneous commodity expressed in terms of weight-quality units. This homogeneous commodity has a unique price per weight-quality unit.

We adopt as a description of the quality trend a function corresponding to the gamma distribution. The function is adjusted so that its maximum value is one.

$$q(x) = dx e^{1-x/\Psi} \quad (20)$$

where d is $1/\Psi$ and Ψ is a parameter. For $q(x)$ maximum,

$$\begin{aligned} \frac{\partial q(x)}{\partial x} &= \frac{q(x)}{x} - \frac{q(x)}{\Psi} \\ &= \left(\frac{1}{x} - \frac{1}{\Psi} \right) q(x) = 0. \end{aligned} \quad (21)$$

Hence, for $q(x) = \text{maximum}$, $x = \Psi$.

This may suggest an estimate of Ψ based on extraneous information. For example, industry and food technology researchers claim that the age preferred by the consumer is nine weeks.⁴ At this age the broiler is considered to be at the best size and quality for broiling. This corresponds to a Ψ of 63, measuring age in days.

To confirm that this is a maximum point, consider the second derivative:

$$\begin{aligned} \frac{\partial^2 q(x)}{\partial x^2} &= -\frac{q(x)}{x^2} + \left(\frac{1}{x} - \frac{1}{\Psi} \right)^2 q(x) \\ &= \left(\frac{1}{\Psi^2} - \frac{2}{x\Psi} \right) q(x) < 0. \end{aligned} \quad (22)$$

As $x < 2\Psi \Rightarrow \partial^2 q(x)/\partial x^2 < 0$, we have a maximum at $x = \Psi$.

Thus, quality has been introduced through adjusting the quantity. For function (8), substitute

$$W^* = f_{ii}(x) q(x). \quad (8.1)$$

This simple device allows us to consider the output of meat (W^*) as a homogeneous commodity.

Decision rules

The conventional assumption of perfect competition is adopted where the producer is a price-taker, and equilibrium is established through the profit-maximization motive.

The individual flock, fixed-product price.—In our derivations we benefit from the fact that each of the relevant production variables is a function of age of bird. Since a flock of a given capacity is considered, maximization of profit per bird is equivalent to maximization of profit per flock. Define

$$\pi_i = P_w \phi_{1i}(x) - P_f \phi_{2i}(x) = R_i - C_i \quad (23)$$

where

π_i = net revenue realized from flock i

ϕ_{1i} = weight as a function of age, where the weight function may take the form (8) or the form (8.1)

⁴ This was suggested in meetings with California industry representatives at Petaluma and staff members from the Department of Food Technology, University of California, Berkeley.

ϕ_{2i} = feed as a function of age, where the feed function may take the form (9) or the form (9.1)

P_w = broiler price⁵ assumed to be invariant between the flocks

P_f = feed price assumed to be invariant between the flocks

R_i = gross revenue from the sale of flock i

and

C_i = cost incurred in raising flock i .

Attention is focused on the feed-weight relation, which is the critical relation in practical management decisions in the broiler industry.

In the form of function here employed, it is common procedure to express the equilibrium conditions in terms of the ratios of costs of given factors of production to value of output. These conditions are summarized in equations (24), considering the several versions previously discussed successively. The reason for following this procedure is, first, methodological, that is, to show the impact of adding dimensions to the description of the growth process and, second, to allow the reader to inspect results for those cases where some of the factors are not considered by the producer to be relevant or important.

Define $S_i = C_i/R_i$. Then, from the first-order conditions we obtain

$$S_i = \frac{\alpha_{1i}}{\beta_{1i}} = \epsilon_i, \text{ based on equations (24.1) (8) and (9)}$$

$$S_i = \frac{\alpha_{1i}}{\beta_{1i} + \gamma x}, \text{ based on equations (8) and (9.1) (24.2a)}$$

$$S_i = \frac{1}{1 + \frac{\gamma}{g_i} x}, \text{ situation (24.2a) under the hypothesis } \alpha_{1i} = \beta_{1i} = g_i \quad (24.2b)$$

$$S_i = \frac{1}{1 + \frac{\gamma}{g_i} x} + \frac{1 - \frac{x}{\bar{\Psi}}}{\gamma + \frac{g_i}{x}}, \text{ based on equations (8.1) and (9.1).} \quad (24.3)$$

Condition (24.1) is derived assuming special relations implying one-to-one correspondence between weight and age and between feed and age, discussed on page 5. In this model a decision about the optimal feed input is *identical* with a decision about the optimal age of marketing and vice versa. The result is that age (representing the time element) vanishes from the decision criterion. As more dimensions are added to the growth process, the time element enters explicitly into the decision criterion through the age variable (x).

In the Israeli economy, breeding is an enterprise carried on by major producers in the broiler industry. It is reasonable to consider these producers (which are the collective Israeli villages called "kibbutzim") as integrated "firms." Hence, it is plausible to assume that management of the broiler industry in a kibbutz might adopt a long-run view where its production coefficients are partially under its control as a result of breeding policy.

The following results may be verified under the assumption $\alpha_{1i} = \beta_{1i} = g_i$:

1. Reducing g results in marketing the flock at an earlier age and thereby profiting from the fact that the feed-weight conversion ratio for a given age is not influenced by the growth rate.

⁵ Because the quality factor is introduced through the growth process, prices received do not change with age.

2. Reduction in γ will result in marketing at a later age since it reduces the total feed consumption for a given age and, hence, increases profitability.
3. An increase in the maximum quality age, Ψ , results in an increase in the optimal age of marketing.

The firm, fixed-product price.—

Up to this stage, opportunity costs of time have been ignored. Recognition of these costs becomes necessary as focus of attention is transferred from the individual flock to the producing firm. This can be done in the previous context simply by redefining π ; in equation (23) to include the net revenue obtained from all flocks raised by a given producer in a given time period.

Consider first the case where all flocks are produced under identical conditions with respect to technical production coefficients, market conditions, and prices. Consider the following simple decision model:

$$\begin{aligned} \pi &= [P_w \phi_1(x) - P_f \phi_2(x)] \frac{K_o}{x} \\ &= (R - C) \frac{K_o}{x} \end{aligned} \quad (23.1)$$

where π , R , and C are defined as in equation (23) and K_o is number of days in the given time period. The index i for flock is omitted because all flocks are identical for a given producer.

The first-order condition is:

$$\frac{d\pi}{dx} = \left(\frac{dR}{dx} - \frac{dC}{dx} \right) \frac{K_o}{x} - \frac{(R - C) K_o}{x^2} = 0.$$

Thus,

$$\frac{dR}{dx} - \frac{dC}{dx} = \frac{R - C}{x}. \quad (25)$$

For an optimal solution, the *marginal* increment in net revenue per flock with respect to x is equal to the *average* increment in net revenue per flock per day. Hence, even in a short period when the effect of interest rate can be neglected, there is an internal rate measuring the opportunity cost of holding the flock an additional day. In this simple case the internal rate is measured by the term on the right of equation (25).

In conclusion, disregarding the problem of financial maturity—that is, disregarding the interest rate—we have brought out the relevance of *waiting* as a factor that has rewards and costs, even in this simple deterministic decision model. Conventional concepts of production input-output relations are not sufficient in this context, as the time element has to be introduced explicitly. Consider now a somewhat more realistic situation where the deterministic decision model is extended to include a firm facing varying prices. Our purpose here is to construct a framework within which the weight-feed relation bears the main influence on the behavior of the producer. This is the case with the non-integrated firm that exists in institutional conditions similar to those of the Israeli economy. However, one may contend that the problem remains relevant in the modern integrated firm, such as in California, where the producer is reduced essentially to a contractual arrangement with a large feed company. In this case the center of decision moves upward, but its nature remains the same. It becomes a suboptimization problem in the general optimization problem of the integrated firm.

The period of analysis is considered short enough to assume capacity to be fixed; and, hence, the factors of production involved in determining capacity may be considered as having negligible

opportunity costs. Note that one can allow for inputs other than feed by appropriate deduction from the price of the broiler.

We adopt here the general approach of Samuelson (1963, pp. 21–36). The fundamental assumption is that the firm tries to maximize its net revenue, and from this the internal conditions of equilibrium are deduced. Additional assumptions are the following:

1. The only endogenous variable is age of flock. All other variables are connected in one-to-one correspondence with age of flock. Since both output and input variables are tied to the same age variable, maximization is derived directly with respect to age.

2. Maximization is with respect to a given period, say a year, and, hence, is subject to the constraint: $\sum_{i=1}^N x_i = K_o$, where N is number of flocks and K_o refers to total available days. Treating the number of flocks (N) as exogenous is justified since orders for renewal flocks are typically placed for annual time intervals. If one wants to allow for intervals between flocks, this can be done by adding a fixed number of days to K_o . Note that one can alter at will the number of days available (K_o) for a given number of flocks (N).

From equation (23) we have net returns per flock i , and total net returns are:

$$\pi = \sum_{i=1}^N [P_{w_i} \phi_{1i}(x) - P_{f_i} \phi_{2i}(x)] \quad (26)$$

$$\begin{vmatrix} (P_{w_1} \phi''_{11} - P_{f_1} \phi''_{21}) & 0 & -1 \\ 0 & (P_{w_2} \phi''_{12} - P_{f_2} \phi''_{22}) & -1 \\ -1 & -1 & 0 \end{vmatrix} > 0$$

where terms are explained in equation (23). The producer maximizes π subject to the constraint:

$$K_o - \sum_{i=1}^N x_i = 0. \quad (27)$$

The Lagrangian to be maximized is:

$$L = \sum_{i=1}^N [P_{w_i} \phi_{1i}(x) - P_{f_i} \phi_{2i}(x)] \quad (28)$$

$$+ \lambda (K_o - \sum x_i).$$

The first-order conditions are:

$$P_{w_i} \phi'_{1i} - P_{f_i} \phi'_{2i} = \lambda \quad i = 1, 2, \dots, N \quad (29)$$

$$K_o - \sum_{i=1}^N x_i = 0$$

where $\phi'_{1i} = \partial \phi_{1i} / \partial x_i$ and $\phi'_{2i} = \partial \phi_{2i} / \partial x_i$.

There are $(N + 1)$ equations and $(N + 1)$ unknowns including N optimal terminal ages and λ . A solution for the system exists, given explicit forms for $\phi_{ji}(x)$, though one may have to resort to iterative solutions. Comparison with the individual flock case, where no time restrictions were introduced, shows that flocks will be marketed at an earlier age when time restrictions are introduced. When the individual flock was considered per se, the marginal increment in net revenue was equated to zero. Here, the marginal increment in net revenue of each flock is equated to the opportunity cost of time (λ). This also equates the marginal increment in net revenue between all flocks. Of course, for L to have a relative maximum, it is necessary that

$$\begin{vmatrix} (P_{w_1} \phi''_{11} - P_{f_1} \phi''_{21}) & 0 & 0 & -1 \\ 0 & (P_{w_1} \phi''_{12} - P_{f_2} \phi''_{22}) & 0 & -1 \\ 0 & 0 & (P_{w_3} \phi''_{13} - P_{f_3} \phi''_{23}) & -1 \\ -1 & -1 & -1 & 0 \end{vmatrix} < 0 \quad (30)$$

$$\dots (-1)^N \begin{vmatrix} (P_{w_1} \phi''_{11} - P_{f_1} \phi''_{21}) & 0 & \dots & 0 & -1 \\ 0 & (P_{w_2} \phi''_{12} - P_{f_2} \phi''_{22}) & \dots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & (P_{w_N} \phi''_{1N} - P_{f_N} \phi''_{2N}) & -1 \\ -1 & -1 & \dots & -1 & 0 \end{vmatrix} > 0$$

where $\phi''_{1i} = \partial \phi'_{1i} / \partial x_i$ and $\phi''_{2i} = \partial \phi'_{2i} / \partial x_i$.

Displacement of equilibrium.—

The set of equations (29) yields an explicit solution for our unknown equilibrium values in terms of the exogenous variables and parameters (for example, prices and production coefficients).

$$x_i^\circ = g^i(Z_1^\circ, \dots, Z_m^\circ) \quad i = 1, 2, \dots, N$$

$$\lambda^\circ = g^{N+1}(Z_1^\circ, \dots, Z_m^\circ)$$

where $Z_1^\circ, \dots, Z_m^\circ$ are the exogenous variables.

Additional information is gained from examining the effects of changes in the exogenous variables. Assume that a firm at an initial equilibrium point, that is, a set x_i° and λ° of the $N + 1$ endogenous variables, is confronted with small changes in the exogenous variables. Here, we shall consider changes only in the price of broilers (P_w) and the total time available (K_o); but similar analyses could be conducted with respect to the price of feed (P_f) or with respect to the growth coefficients (g_i, γ, Ψ). The response of the producer is derived then by solving for the slopes $\partial x_i / \partial Z_j$ in the

$$\text{equation } dx_i = \sum_{j=1}^m (\partial x_i / \partial Z_j) dZ_j \quad (i = 1,$$

$$2, \dots, N), \text{ and } d\lambda = \sum_{j=1}^m (\partial \lambda / \partial Z_j) dZ_j.$$

To obtain this, take the total differential of each of the $(N + 1)$ implicit functions of the first-order conditions:

$$d(P_{w_i} \phi'_{(1i)^\circ} - P_{f_i} \phi'_{(2i)^\circ} - \lambda^\circ)$$

$$= (P_{w_i} \phi''_{1i} - P_{f_i} \phi''_{2i})$$

$$dx_i - d\lambda + \phi'_{(1i)^\circ} dP_{w_i} = 0$$

$$i = 1, 2, \dots, N$$

and

$$\begin{aligned} d(K_o - \sum_{i=1}^N x_i) \\ = -dx_1 - dx_2 \dots - dx_N + dK_o = 0 \end{aligned} \quad (31)$$

where the () subscript denotes the fact that we start from an initial equilibrium point. We assume no change in the remaining parameters, that is, $dP_{f_i} = dg_i = \dots = dZ_j = 0$.

Equation (31) may be written in matrix form:

$$\begin{bmatrix} (P_{w_1} \phi''_{11} - P_{f_1} \phi''_{21}) & 0 & \cdots & 0 & -1 \\ 0 & (P_{w_2} \phi''_{12} - P_{f_2} \phi''_{22}) & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & (P_{w_N} \phi''_{1N} - P_{f_N} \phi''_{2N}) & -1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_N \\ d\lambda \end{bmatrix} + \quad (32)$$

$$\begin{bmatrix} \phi'_{(11)\circ} & 0 & \cdots & 0 & 0 \\ 0 & \phi'_{(12)\circ} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \phi'_{(1N)\circ} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} dP_{w_1} \\ dP_{w_2} \\ \vdots \\ dP_{w_N} \\ dK_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

or in partitioned matrix form:

$$\left[\begin{array}{c|c} H & -L \\ \hline -L' & 0 \end{array} \right] \begin{bmatrix} dx_j \\ d\lambda \end{bmatrix} + \left[\begin{array}{c|c} \phi'_{(1)\circ} & 0 \\ \hline 0 & 1 \end{array} \right] \begin{bmatrix} dP_{w_j} \\ dK_o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (33)$$

where

From equation (33) we obtain

$H = (N \times N)$ matrix

$\phi'_{(1)\circ} = (N \times N)$ matrix

$L =$ an N dimensional column vector

$dx_j =$ an N dimensional column vector ($j = 1, 2, \dots, N$)

$dP_{w_j} =$ an N dimensional column vector ($j = 1, 2, \dots, N$)

$d\lambda =$ scalar

and

$dK_o =$ scalar.

$$\begin{bmatrix} dx_j \\ d\lambda \end{bmatrix} = \left[\begin{array}{c|c} H & -L \\ \hline -L' & 0 \end{array} \right]^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\phi'_{(1)\circ} & 0 \\ \hline 0 & -1 \end{bmatrix} \begin{bmatrix} dP_{w_j} \\ dK_o \end{bmatrix}$$

Denote

$$A = \left[\begin{array}{c|c} H & -L \\ \hline -L' & 0 \end{array} \right]$$

Then, for A to have an inverse, $|A| \neq 0$.

We note that $|A|$ is the N th bordered Hessian, which assures that there exists a unique solution for the maximization problem different from zero. Intuitively, it is obvious that this condition is closely connected with inter- and intraseasonal

price variation. As it turns out A^{-1} (the inverse of A) is of relatively simple and consistent pattern, and it is possible to derive a general form for the case of N flocks.

Denote:

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 & -1 \\ 0 & a_2 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & & a_N & -1 \\ -1 & \dots & \dots & -1 & 0 \end{bmatrix} \text{ and in partitioned form, } A^{-1} = \frac{1}{\Delta} \begin{bmatrix} B & | & b \\ \hline b' & | & c \end{bmatrix}. \quad (35)$$

The general formula for Δ can be shown to be:

$$\Delta = |A| = - \sum_{i=1}^N a_{i1} a_{i2} \dots a_{iN-1} \quad (36)$$

where the summation is over all N combinations of $(N - 1)$ out of $\{1, 2, \dots, N\}$. Alternatively,

$$\Delta = - \left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_N} \right) (a_1 a_2 \dots a_N) \quad (36.1)$$

$$(a_1 a_2 \dots a_N) = - \left(\prod_{k=1}^N a_k \right) \left(\sum_{i=1}^N \frac{1}{a_i} \right).$$

$B = \{b_{ij}\}$ is a symmetrical matrix of order N with:

$$b_{ii} = - \frac{1}{a_i} \left(\prod_{k=1}^N a_k \right) \left(\sum_{j \neq i}^N \frac{1}{a_j} \right) \quad (37)$$

$$b_{ij} = b_{ji} = \frac{\prod_{k=1}^N a_k}{a_i a_j} \Rightarrow b_{ii} = \sum_{j \neq i}^N b_{ij}.$$

b is a column of order N whose i th element is

$$b_i = \frac{\prod_{k=1}^N a_k}{a_i} \quad (38)$$

and the scalar c is

$$c = \prod_{k=1}^N a_k \Rightarrow b_i = \frac{c}{a_i}. \quad (39)$$

Now, equation (34) may be written in the form:

$$\begin{bmatrix} x_{P_w} & | & x_{K_o} \\ \hline \lambda'_{P_w} & | & \lambda_{K_o} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} -B\phi'_{(1)} & | & -b \\ \hline -b'\phi'_{(1)} & | & -c \end{bmatrix} \quad (40)$$

where the left-hand side is a matrix of rates of change of the endogenous variables $(x_1, \dots, x_N, \lambda)$ with respect to the exogenous variables $(P_{w1}, \dots, P_{wN}, K_o)$.

Consider, first, the special case of two flocks ($N = 2$). Define

$$a_i = P_{w_i} \phi''_{1i} - P_{f_i} \phi''_{2i}, \quad i = 1, 2.$$

Then the bordered Hessian is:

$$\begin{vmatrix} a_1 & 0 & -1 \\ 0 & a_2 & -1 \\ -1 & -1 & 0 \end{vmatrix} = -(a_1 + a_2) > 0. \quad (41)$$

And the relevant slopes

$$\begin{bmatrix} \frac{\partial x_1}{\partial P_{w_1}} & \frac{\partial x_1}{\partial P_{w_2}} & \frac{\partial x_1}{\partial K_o} \\ \frac{\partial x_2}{\partial P_{w_1}} & \frac{\partial x_2}{\partial P_{w_2}} & \frac{\partial x_2}{\partial K_o} \\ \frac{\partial \lambda}{\partial P_{w_1}} & \frac{\partial \lambda}{\partial P_{w_2}} & \frac{\partial \lambda}{\partial K_o} \end{bmatrix} = -\frac{1}{a_1 + a_2} \begin{bmatrix} \phi'_{(11)\circ} & -\phi'_{(12)\circ} & -a_2 \\ -\phi'_{(11)\circ} & \phi'_{(12)\circ} & -a_1 \\ -a_2\phi'_{(11)\circ} & -a_1\phi'_{(12)\circ} & -a_1a_2 \end{bmatrix} \quad (42)$$

In particular

$$\begin{aligned} \frac{\partial x_1}{\partial P_{w_1}} &= -\frac{\phi'_{(11)\circ}}{a_1 + a_2} > 0 \\ \frac{\partial x_2}{\partial P_{w_2}} &= -\frac{\phi'_{(12)\circ}}{a_1 + a_2} > 0 \\ \frac{\partial x_1}{\partial P_{w_2}} &= \frac{\phi'_{(12)\circ}}{a_1 + a_2} < 0 \\ \frac{\partial x_2}{\partial P_{w_1}} &= \frac{\phi'_{(11)\circ}}{a_1 + a_2} < 0 \end{aligned} \quad (43)$$

since from equation (41), $-(a_1 + a_2) > 0$, and the marginal growth with respect to age (ϕ'_{1i}) is always positive.

An increase in the price of flock i postpones the date of marketing the flock while reducing by an equivalent amount the days remaining for the other flock. Hence, the rate of substitution between x_1 and x_2 for $dP_{w_1} = dP_{w_2} = dK_o = 0$ is

$$\frac{dx_1}{dx_2} = \frac{\frac{\partial x_1}{\partial P_{w_1}}}{\frac{\partial x_2}{\partial P_{w_1}}} = \frac{\frac{\partial x_1}{\partial P_{w_2}}}{\frac{\partial x_2}{\partial P_{w_2}}} = -1. \quad (44)$$

The rates of change of x_i with respect to an increase in the time constraint K_o are

$$\frac{\partial x_1}{\partial K_o} = \frac{a_2}{a_1 + a_2} \text{ and } \frac{\partial x_2}{\partial K_o} = \frac{a_1}{a_1 + a_2}. \quad (45)$$

Hence, a necessary and sufficient condition for $\partial x_i/\partial K_o > 0$ is that for flock i' , $a_{i'} < 0$, $i' \neq i$, and $i, i' = 1, 2$.

In the case where the quality factor is ignored, we have, in the relevant range of the weight and feed functions, $\phi''_{1i} < 0$ and $\phi''_{2i} < 0$. Therefore, $a_{i'} < 0$ implies that

$$|P_{w_{i'}} \phi''_{1i'}| > |P_{f_{i'}} \phi''_{2i'}|. \quad (46)$$

Under the hypothesis $\alpha_{1i} = \beta_{1i} = g_i$, the inflection point of weight with respect to age will be reached before the inflection point of feed with respect to age (see page 5). This implies that condition (46) will hold.

Under the same assumptions we obtain:

$$\frac{\partial \lambda}{\partial P_{w_i}} = \frac{\partial x_i}{\partial K_o} \phi'_{(1i)\circ} > 0$$

and (47)

$$\frac{\partial \lambda}{\partial K_o} = \frac{a_1 a_2}{a_1 + a_2} < 0, \quad i = 1, 2.$$

For a given number of flocks *per period* K_o , changing K_o will determine the number of flocks *per year*. Attaching the conventional interpretation to λ , that is, a shadow price for the constraint K_o ,

note that λ is positively related to P_{w_i} and negatively to K_o .

The case of two flocks is a special example introduced for simplicity. For the general case of N flocks, we obtain from

$$\frac{\partial x_i}{\partial P_{w_i}} = -\frac{b_{ii} \phi'_{(1i)} \circ}{\Delta} = -\frac{\phi'_{(1i)} \circ}{a_i} \left[1 - \frac{1}{a_i \sum_{i=1}^N \frac{1}{a_i}} \right]. \quad (48)$$

$$\frac{\partial x_i}{\partial P_{w_j}} = -\frac{b_{ij} \phi'_{(1j)} \circ}{\Delta} = \frac{\phi'_{(1j)} \circ}{a_j} \frac{1}{a_i \sum_{i=1}^N \frac{1}{a_i}}. \quad (49)$$

$$\frac{\partial x_i}{\partial K_o} = -\frac{b_i}{\Delta} = \frac{1}{a_i \sum_{i=1}^N \frac{1}{a_i}}. \quad (50)$$

$$\frac{\partial \lambda}{\partial P_{w_i}} = -\frac{b_i \phi'_{(1i)} \circ}{\Delta} = \frac{\phi'_{(1i)} \circ}{a_i \sum_{i=1}^N \frac{1}{a_i}}. \quad (51)$$

$$\frac{\partial \lambda}{\partial K_o} = -\frac{C}{\Delta} = \frac{1}{\sum_{i=1}^N \frac{1}{a_i}}. \quad (52)$$

These results are of use only if something can be learned from them about the direction and boundaries of the rates of change. It can be established that $\partial x_i / \partial P_{w_i} > 0$. To verify this, note that $b_{ii} = \Delta_{(i)}$ —the bordered Hessian of order N (excluding flock i)—will always be of opposite sign to Δ , which is of order $N + 1$. Hence, $-b_{ii} / \Delta > 0$ and $\phi'_{(1i)} \circ > 0 \Rightarrow \partial x_i / \partial P_{w_i} > 0$.

This result implies that a seasonal rise in price (at a certain month, say, December) results in a reduction in the supply of broiler meat to the market. This, in turn, increases the excess of demand over supply in the market and, hence, causes a further increase in the

the set of equations (40) the following expressions for the corresponding rates of change in the endogenous variables with respect to small changes in the corresponding exogenous variables:

price of broilers. This, together with the lag for hatching additional flocks, could account for the short-run price instability typical of the broiler industry.

To evaluate the slopes defined by equations (49) to (52) impose the condition (46) for all i ($i = 1, 2, \dots, N$)—that is, all $a_i < 0$. As a direct result, it can be verified that

$$\frac{\partial x_i}{\partial P_{w_j}} < 0; \frac{\partial x_i}{\partial K_o} > 0; \frac{\partial \lambda}{\partial P_{w_i}} > 0;$$

$$\frac{\partial \lambda}{\partial K_o} < 0; \frac{\partial x_i}{\partial x_j} < 0$$

which conforms to the results obtained for the case $N = 2$. For $\partial x_i / \partial K_o$ we can

also establish an upper bound. From equations (48) and (50) we obtain:

$$\frac{\partial x_i}{\partial P_{w_i}} = - \frac{\phi'_{(12)_i}}{a_i} \left(1 - \frac{\partial x_i}{\partial K_o} \right).$$

Since $a_i < 0$, it follows that

$$\frac{\partial x_i}{\partial P_{w_i}} > 0 \Rightarrow 0 < \frac{\partial x_i}{\partial K_o} < 1.$$

From this analysis it appears that,

under relatively few and reasonable assumptions, important information may be revealed about optimal producer behavior. This qualitative information can be transformed into quantitative information by introducing explicit production relations. The investigation might also be extended to derive rates of change with respect to parameters that can be influenced directly by the policy-maker or breeder—in particular, the price of feed and the production coefficients.

A Sequential Stochastic Decision Model

In the previous section growth functions relating weight, feed, and quality to age of the flock were developed. Then, decision rules were applied in a deterministic context to a firm confronted with such a set of growth functions. The solutions required searching for the optimal marketing age for a commodity undergoing a growth process. The case of a single flock and cases of N flocks under constant and varying prices were considered. In this approach the time element was incorporated in the production process by expressing each of the production variables as a function of age. One must recognize, however, that, by staying within a deterministic analytical framework, important aspects of the decision problem are ignored. Hence, the analysis is extended in this section to a sequential stochastic decision framework.

The producer enters the period, say, a given week, with a stock of a "living" commodity (which could be livestock, timber, field crops, wine, etc.), given the current price and a probability distribution of prices for the following week. He must decide whether to sell his inventory this week and buy a stock

of a younger vintage or to keep the existing stock until the next week.

The features characterizing the problem are:

1. The time element enters at two levels: (a) The stock in hand is undergoing a growth process and (b) the decision process itself is executed over time. The previous section only recognizes explicitly the first level.
2. Because the decision concerns activities in future time and the decision process is implemented at specified future time intervals, there is risk involved. Even assuming that the physical growth process can be controlled by the producer, risk remains because the individual firm cannot control prices.
3. The decision at each stage determines whether transition to the next period will be with the old stock or with a new one. This property of the problem prevents us from accepting the transition probability matrix as exogenous.

Several authors have applied this ap-

proach to the problem of replacement in the specific framework of a Howardian model (Howard, 1960; Giaever, 1966; and Ward, 1964). The Howardian model is based on the Markov process as a system model and uses dynamic programming as its method of optimization. In this general framework, the Howardian model specifies for each action a transition probability matrix where the transition bears a given reward. Howard's automobile replacement model is a relevant example. The model is characterized by an exogenous transition probability, measuring the probability that a car of age i will survive to age $i + 1$ without incurring a prohibitively expensive repair.

As noted on page 2, Burt (1965) uses the Howardian model in a special case for which an analytic solution is possible. The problem solved by Burt gives the optimal age of replacement of an asset under conditions of chance failure or loss. Hence, the endogenous decision variable, whose value is to be determined, is optimal age, while the transition probabilities are exogenous to the economic model. The Howardian model does not suffice for the problem under consideration in this section. Nevertheless, our model, like the Howardian model, is based on a Markovian process. Similarities and dissimilarities with the Howardian model are noted subsequently.

Our development is with specific reference to the broiler producer in Israel. The nature of the operation was described in the previous section. The following initial assumptions are adopted:

1. Replacement of the flock by a new

one is instantaneous. Usually, one has to allow a few days for preparation, which can be easily allowed for. However, the nature of the solution can be presented in this simplified form. Later, the effect of relaxing this simplifying assumption is examined.

2. Age of the new flock (replacing the flock sold) is to be always at six weeks. This is the typical age at which broilers are transferred from nursery to barn.⁶
3. The producer operates at a given capacity. Therefore, maximization can be placed on a per-bird basis.
4. The major factors in production are those recognized in the previous section (weight, feed, and quality). As was pointed out, the importance of these factors remains under different institutional conditions, even though the decision center may move from the individual producer upward to an integrated company.

The solution entails a search for the optimal decision-making rules that will give the vector of cutoff prices (the critical values) which determines (all other things equal) at what current prices the producer will keep the growing inventory until the next period or will sell and replace with new stock. This solution differs from the one sought by Burt (1965) and Giaever (1966), which is for the optimal age of replacing a fixed asset bearing an annual net return.

Two methods of solution are considered in the case of the homogeneous model.⁷

⁶ The problem can be modified to include the case of replacement by one-day-old chicks; we shall discuss this modification later in this section.

⁷ In the homogeneous model the probability distribution of prices is assumed to be the same in all weeks.

1. The functional equation method developed by Bellman (Bellman and Dreyfus, 1962).
2. The analytic solution.

The functional equation solution

Solution of the optimization problem consists of solving for $(X - 7)$ cutoff prices $(P^*(7), P^*(8), \dots, P^*(X - 1))$. Thus, the optimal policy consists of finding a unique vector of critical values of prices that will determine, for each age, the price below which the producer will keep the flock for another week and above which he will sell the flock during the current week. (To simplify, it is assumed that all transactions take place at the beginning of the week, say, Monday.)

The method of solution suggested here employs dynamic programming techniques (Bellman and Dreyfus, 1962). The state of the system is defined by age of flock. That is, growth functions relate weight, quality, and feed consumption to age. Because of technological achievements in the broiler industry, it is assumed that the producer can accurately predict, within the boundaries of economic significance, the empirical growth functions from one week to another.

Prediction of market price of broilers is subject to wider margins of error. Though current market prices are known, prices next week are not known; and important information is lost by introducing prices for next week in a de-

terministic way. Accordingly, broiler prices are assumed to be a random variable having, in the homogeneous model, the same distribution $h(P)$ over all weeks. These prices are independent of age of broiler since weight is "corrected" for quality changes through the quality function. P_n is the current market price from the density function $h(P)$. Price of feed is assumed to be given, which is the case in Israel, where feed price is controlled.

Define now a gross return function for each current price P_n , $R(x, P_n) = W(x)P_n$, which measures gross returns per bird at age x and price P_n . Define further the cost function, $C(x) = C_0 + P_f F(x)$, where C_0 measures fixed costs and replacement costs, while $P_f F(x)$ measures cost of feed per bird up to age x .⁸ Thus, the immediate net income for a flock at a given age and a current market price can be defined as $R(x, P_n) - C(x)$.

The formal solution may be described briefly as follows: Given a set of states, $S = s(x)$, and a set of actions, $A = \{KR\}$,⁹ map $S \xrightarrow{D} A$, where D is the decision rule used in mapping S on A —that is, $A = D(S)$. Let $f_n(x)$ represent the maximum expected return for the last n periods if the producer begins at stage n^{10} with a flock at age x . Then,

$$f_n(x) = \max_{P^*(x)} \quad (53)$$

$$\left\{ \int_0^{\infty} r(P_n, P_n^*(x), x) h(P) dP \right\}$$

⁸ The cost of money could also be taken into account through a discounting factor $\beta = 1/1 + r$, where r is the interest rate per week. Because broiler production is a short-cycle operation, β will be close to 1—that is, r will be small. For example, the discount factor per week is $\beta = .9925$ for an interest rate of 12 per cent per year. Nevertheless, the introduction of a discount rate might be justified if one were looking into an infinite horizon for an operation which commits a considerable amount of working capital that could be invested elsewhere.

⁹ K means keep the flock for another week and R denotes sell and replace with a new flock.

¹⁰ Stage n denotes the n th period from the end of the horizon.

where

$$r(P_n, P_n^*(x), x) = \begin{cases} f_{n-1}(x+1) & \text{if } P_n < P_n^*(x) \\ R(x, P_n) - C(x) + f_{n-1}(7) & \text{if } P_n \geq P_n^*(x) \end{cases}$$

for $x = 7, 8, \dots, X - 1$

X = termination age at which the flock will be sold, whatever the price

and

$P_n^*(x)$ = cutoff price at age x for the n th stage.

Substituting for $r(P_n, P_n^*(x), x)$, (53) may be written:¹¹

$$f_n(x) = \max_{H_x} \left\{ H_x f_{n-1}(x+1) + (1 - H_x) \right. \quad (54)$$

$$\left. \left[\int_{P_n^*(x)}^{\infty} R(x, P_n) \frac{h(P)}{1 - H_x} dP - C(x) + f_{n-1}(7) \right] \right\}$$

where $H_x = \int_0^{P_n^*(x)} h(P) dP$, for $x = 7, 8, \dots, X$.

Each strategy chosen by the producer determines simultaneously the cutoff price $P_n^*(x)$, the transition probabilities H_x and $(1 - H_x)$, and the immediate reward. Moreover, because H_x is connected in one-to-one correspondence with the cutoff price, at each age one and only one decision variable determines the strategy taken.

The transition probability matrix has the following form:

$$T = \begin{bmatrix} 1 - H_7 & H_7 & 0 \cdots 0 \\ 1 - H_8 & 0 & H_8 \cdots 0 \\ \vdots & \vdots & \vdots \\ 1 - H_{X-1} & 0 & 0 \cdots H_{X-1} \\ 1 & 0 & 0 \cdots 0 \end{bmatrix} \quad (55)$$

T is a transition probability matrix describing a completely ergodic Markov chain (Howard, 1960).¹² An important property of this model is that the transition matrix is endogenous. This is not the case in the Markovian dynamic models of Howard (1960) and Burt (1965) (see page 2). Hence, methods of solving for the optimal policy in the case of an exogenous Markov transition probability matrix can be used here only through the use of approximations.

To obtain a numerical solution for (54), the continuous prices are approximated by a discrete array. A convenient way is to break $h(P)$ into a histogram of K equal probability rectangles. This results in an array of K prices and an associated set of discrete probabilities. Empirical results are presented in the section beginning on page 31. The process converges rapidly on the optimal vector of cutoff prices (Howard, 1960). The approximative nature of the solution should not be considered a serious limitation to the usefulness of the numerical solution. Nevertheless, one does lose insight into the analytic structure of the model that provides important links to the economics of the firm. On the other hand, as noted more fully later, relaxing some of the simplifying assumptions underlying the homogeneous model may make the numerical solution of the approximative functional equation approach more attractive.

¹¹ To simplify notation, the subscript n is omitted from $P_n^*(x)$ and H_{nx} .

¹² "We shall designate as a completely ergodic process any process whose limiting state probability distribution is independent of starting conditions" (Howard, 1960, p. 6).

The analytic nature of the solution, its existence, convergence, and economic meaning are considered subsequently. However, before diverting attention in this direction, we shall investigate the properties of the Markovian decision process—especially the properties of the equilibrium state probabilities. In doing so we use results obtained for the Howard model (Burt, 1965) because it is at this point that the two models meet.

The solution for the optimal policy defines a unique transition probability T^* . Regarding T^* as a transition prob-

ability matrix for an ergodic chain, the vector of steady state probabilities can be derived from the set of equations.

$$q = q T^* \tag{56}$$

subject to

$$\sum_{x=7}^x q_x = 1, \quad q_x \geq 0 \text{ for all } x \tag{57}$$

where q is the vector of steady state probabilities.

It can be shown that solution of (56) subject to constraint (57) gives

$$\begin{aligned} q_7 &= \frac{1}{1 + H_7 + H_7H_8 + \dots + H_7H_8 \dots H_{X-1}} = \frac{1}{D}; \\ q_8 &= \frac{H_7}{D} = H_7q_7; \quad q_9 = \frac{H_7H_8}{D} = H_8q_8; \dots; \\ q_X &= \frac{H_7H_8 \dots H_{X-1}}{D} = H_{X-1}q_{X-1}. \end{aligned} \tag{58}$$

The vector q measures the equilibrium probabilities of a flock being at age x , if the system is allowed to approach an infinite number of stages.

Now, the average return of the system as it operates for a long period may be defined as a weighted average net revenue over all possible ages of the flock, using the steady state probabilities of being at these ages as weights. Accordingly,

$$\begin{aligned} \pi &= \sum_{x=1}^x [m(P^*(x), x) \\ &\quad - (1 - H_x) C(x)] q_x = \sum_{x=1}^x E(x) q_x \end{aligned} \tag{59}$$

where

$$m(P^*(x), x) = \int_{P^*(x)}^{\infty} R(x, P) h(P) dP.$$

These results provide a method of computing values of state probabilities or expected returns corresponding to the equilibrium toward which the system will progress in the future.¹³ However, these results, although shedding additional light on the statistical properties of the decision process, depend upon having a prior solution for the critical set H_x . The next section is devoted to deriving an analytic framework for the solution of H_x , based on application of the decision rules indicated above.

The analytic solution

The general idea of the analytic solution can be presented in the following way: From equation (54) it can be seen that at each age a decision is made either to sell this week or to keep until

¹³ In its essence, the equilibrium is still one of a short-run nature because the factors operating in the system remain of a short-run nature even though the economic horizon is infinite.

next week. In general, the following set of equations describe the optimal decision rules for the corresponding age groups:

$$V(x) = \max \left\{ P(x) W(x) - C(x), \int_{P^{*(x+1)}}^{\infty} V(x+1) h(P) dP - \pi \right\} \tag{60}$$

for $x = X - 1, \dots, 7$

where

$$V(X) = W(X) \int_0^{\infty} P h(P) dP - C(X), \text{ since } H_X = 0$$

$$W(x) = \text{weight of flock at age } x^{14}$$

$$P(x) = \text{cutoff price at age } x$$

$$C(x) = \text{cost of keeping the flock until age } x$$

and

π = expected returns of one week, as the system operates for a long period and each age group has a certain probability to appear.

π must be deducted because this measures the average opportunity costs of postponing sale of the flock for another week.

The set of equations (60) describes rational behavior on the part of a pro-

ducer motivated by profit maximization who decides at each age whether to sell the flock at age x or to keep it. The producer will sell the flock if the immediate realized net return ($P_n W_x - C_x$) is greater than the expected net return over the remaining period (from age x to X). He will keep the flock if immediate returns are smaller than expected returns. The recursive nature of the decision process means that, starting from the termination age X , the producer is concerned at each stage only with ages greater than the age under consideration. The reason is intuitively obvious because at age x_0 the decision to keep would have already been made for any $x < x_0$.¹⁵ Equations (60) are useful in describing the solution procedure for the decision process. But, before turning to the properties of the system (that is, convergence), let us remain for the moment in the context of equilibrium and state the optimal policy derivable from the decision process described in (60).

The optimal policy will define a unique vector of cutoff prices ($P^*(7), P^*(8), \dots, P^*(X - 1)$) such that

$$P^*(x) W(x) - C(x) = E(x+1) - \pi + H_{x+1}(E(x+2) - \pi) + \dots + H_{x+1} H_{x+2} \dots H_{X-1} (E(X) - \pi) \tag{61}$$

for $x = X - 1, X - 2, \dots, 8, 7$

where, using (60),

$$E(X) = W(X) \int_0^{\infty} P h(P) dP - C(X), \text{ given } H_X = 0. \tag{62}$$

$$E(x) = W(x) \int_{P^{*(x)}}^{\infty} P h(P) dP - (1 - H_x) C(x), \text{ given } H_x = \int_0^{P^{*(x)}} h(p) dP$$

for $x = X - 1, X - 2, \dots, 8, 7$.

¹⁴ Weight is measured in "modified" units to include both quantitative and qualitative growth.

¹⁵ This intuitive result is based on the concavity properties of the profit function in the deterministic case.

Substituting (62) in (61) forms a system of $(X - 7)$ equations with $(X - 6)$ unknowns. An additional equation is

needed to make the system self-contained. The additional equation is that defining the expected return per week:

$$\pi = \frac{E(7) + H_7 E(8) + H_7 H_8 E(9) + \dots + H_7 H_8 H_9 \dots H_{X-1} E(X)}{(1 - H_7) + 2(1 - H_8) H_7 + 3(1 - H_9) H_8 H_7 + \dots + (X - 7) H_{X-1} H_{X-2} \dots H_8 H_7} \quad (63)$$

where the numerator measures the expected return per flock up to age X , and the denominator measures the expected life of the flock. After simple algebraic manipulation, the denominator may be written

$$D = 1 + H_7 + H_7 H_8 + \dots + H_7 H_8 H_9 \dots H_{X-1}. \quad (64)$$

Hence, equation (63) is identical to equation (59) where the q 's are as defined in (58) and the E 's are as defined both in (59) and in (62).

The optimal policy (defined by the vector of cutoff prices $P^*(7), P^*(8), \dots, P^*(X - 1)$) and the corresponding expected net returns per week, π^* , are determined simultaneously by the set of equations (61) and equation (63), using the definitions in (62).

The same solution can be derived from a different angle, giving more insight into the problem. It was noted above that, given the transition probability matrix T (expression 56), one can derive the steady state probabilities (q_x) and from there define π (equation 59). The main difficulty in deriving the solution using the transition probability matrix as a basis (the Howard model) is that in our case the matrix T is endogenous. The H 's are the decision variables, being tied to the cutoff prices in a one-to-one correspondence. However, one can derive the analytic solution starting from the assumption that the system is in the steady state, that is, assuming that equation (59) holds. What is involved, then, is to

maximize π with respect to H_7, H_8, \dots, H_{X-1} . This is the same as maximizing π with respect to $P^*(7), P^*(8), \dots, P^*(X - 1)$.

The first-order conditions for maximizing π in equation (59) are:

$$\frac{\partial \pi}{\partial H_\nu} = \sum_{x=i}^X E(x) \frac{\partial q_x}{\partial H_\nu} + C(\nu) q_\nu + \frac{\partial m(P^*(\nu), \nu)}{\partial P^*(\nu)} \frac{\partial P^*(\nu)}{\partial H_\nu} q_\nu = 0 \quad (65)$$

for $\nu = 7, 8, \dots, X - 1$. Upon expansion and rearranging terms, (65) becomes:

$$P^*(\nu) W(\nu) - C(\nu) = \sum_{j=\nu+1}^X [E(j) - \pi] \frac{q_j}{q_{\nu+1}} \quad (66)$$

for $\nu = 7, 8, \dots, X - 1$.

The left-hand side of (66) measures the *opportunity cost* of a decision to keep the flock at age ν , evaluated at the corresponding *cutoff price*. The right-hand side measures the *conditional expected increment* in net return given that the decision is to keep the stock at age ν .

The set of equations (66) can be reduced to the following set of *difference equations*:

$$P^*(\nu) W(\nu) - C(\nu) = E(\nu + 1) - \pi + H_{\nu+1} [P^*(\nu + 1) W(\nu + 1) - C(\nu + 1)] \quad (67)$$

for $\nu = 7, 8, \dots, X - 1$.

A negative relation between cutoff prices and age of flock would be expected intuitively based on the following argu-

ment: In the deterministic case considered earlier, the optimal age of marketing is determined uniquely. One of the results obtained there was that an *increase* in output price, other things equal, implies a *reduction* in the optimal age of marketing. From this result one may deduce that, in the stochastic case, if at age x the producer's decision is to sell the flock at any price equal to or greater than $P^*(x)$, this range of prices would be expected to remain in the "sell domain" at age $(x + 1)$. And because the profit function in the deterministic case has the property of diminishing marginal profit in the neighborhood of the optimal marketing age, the sell domain will, in general, be an increasing function of age. This means that $P^*(x)$ is a decreasing function of age.

From equations (61)¹⁶ and (63), an iterative solution can be derived relatively easily using a desk calculator. Assume that, as a first approximation, all cutoff prices are set at the average price ($P_0(x) = \int_0^\infty P h(P) dP$, for all x). Assuming either a normal or a uniform distribution of prices, the corresponding H_x is $1/2$ for all x . This furnishes an initial π^0 for computing a corresponding new set of H_x^1 (or equivalently, $P_1(x)$). This, in turn, results in π^1 , which is the basis for the next approximation, etc.

An empirical solution is derived in the section starting on page 31. It is shown there that convergence is fast, and the results are identical to those of Bellman's method. The computations might be reduced by assigning initial cutoff prices corresponding more closely to behavior in the real world. For example, higher cutoff prices might be assigned to early ages and lower cutoff prices to later ages. It remains to be shown that following this procedure always con-

verges to the optimal policy and the optimal π .

The system of equations (61) and (63) defines a unique solution $\pi(P^*(x))$ since it satisfies the first-order conditions in (66). Nevertheless, the optimal $\pi(P^*(x))$ cannot be solved for directly and, hence, one has to start from a suboptimal policy corresponding to a suboptimal π . In this context, convergence of the system becomes crucial.

In showing convergence of the system, the following properties of the model are used:

1. *Dichotomy of the System.* The optimal policy $P^*(x)$ is defined by the set of equations (61), given π^* ; while π^* is defined by equation (63), given $P^*(x)$. This is a direct result of the ergodic properties of the Markovian decision process.
2. *Recursiveness of the System.* As described by (61), the policy depends only on decisions based on current and older ages but not on earlier ages of the flock.

Assume that the desired policy is $P_0(x)$, each element of $P_0(x)$ being equal to the expected price. The corresponding π is obtained from (63), and π^0 is used as an initial value in the search for an optimal policy. Obviously, $\pi^0 \leq \pi^*$, since π^* is the maximum that can be achieved from all possible policies. For a given $\pi^0 = \pi^* - \epsilon$, ($\epsilon \geq 0$) apply decision rule (61) which generates a new vector of cutoff prices $P_1(x)$. The new vector $P_1(x)$ defines a new π^1 . By the same reasoning as above, $\pi^1 \leq \pi^*$. But in this case there is also a lower bound defined by the inequality $\pi^1 \geq \pi^0$. This can be verified from (61). Cutoff prices are determined recursively that equate realized profit at the cutoff points to the corresponding

¹⁶ Equation (67) may be substituted for (61) because it is more convenient for computation.

expected returns normalized for given π^0 . But since $\pi^0 < \pi^*$, following the optimal decision rule, actual profit at the end of the cycle will be greater than π^0 . The new π^1 is used to derive a new set $P_2(x)$ that defines π^2 ; $\pi^1 \leq \pi^2 \leq \pi^*$. Iteration continues until $\pi^{n-1} = \pi^n = \pi^*$.

Extensions of the homogeneous model

In the following some of the simplifying assumptions are relaxed.

Starting with one-day-old chicks.

—The homogeneous model assumes two stages in production: The flock is kept at the nursery up to an age of six weeks and then transferred into the barn. The decision model is applied to the production process in the barn, always assuming instantaneous transition to new flocks at age six. In the real production process, a few days should be allowed for

preparing the barn for a new flock. Moreover, it is common practice at a large number of farms in Israel (and in the United States) to place new flocks in the barn at an age of one day, supplying nursery services at the barn. Through this procedure, the stress of transferring the flock at six weeks is avoided. These modifications can be easily introduced into the model.

Assume that the process starts with a one-day-old flock, but the earliest age of sale remains at seven weeks. This change is accommodated by modifying the term D in equation (64). Write the new D term as:

$$D_1 = 7 + H_7 + H_7H_8 + \dots + H_7H_8 \dots H_{X-1}, \tag{68}$$

which incorporates the constraint $H_1 = H_2 = \dots = H_6 = 1$. Then,

$$\pi_1 = \frac{E_7 + H_7 E(8) + \dots + H_7H_8 \dots H_{X-1} E(X)}{D_1}. \tag{69}$$

The denominator is the same as in equation (63) except that $D_1 > D$, since $D_1 - D = 6$. Hence, $\pi_1 < \pi$. This, in turn, will raise the equilibrium cutoff prices at all ages (7 to $X - 1$).

The D term is adjusted in a similar way to allow for the two weeks needed for preparing the barn. In this case, 3 would replace 7 as the first right-hand term in equation (68).

Interseasonal model.—The influence of seasonality can be introduced in a fairly simple way. As Bellman and Dreyfus (1962, pp. 118–23) show in a different context, the return functions in this case will depend on calendar date of the stage as well as on age of flock. To accommodate this, rewrite (53) as follows:

$$f_n(x, t) = \max_{P^*(x, t)} \tag{70}$$

$$\left\{ \int_0^\infty r(P_n, P^*(x, t), x, t) h(P)_t dP \right\}$$

where

$$r(P_n, P^*(x, t), x, t) = \begin{cases} f_{n-1}(x + 1, t + 1) & \text{if } P_n < P^*(x, t) \\ R(x, t, P_n) - C(x, t) + f_{n-1}(7, t + 1) & \text{if } P_n \geq P^*(x, t) \end{cases}$$

for $x = 7, 8, \dots, X - 1$; $t = 1, 2, \dots, 52$ weeks.

The most important seasonal factor would seem to be seasonal variation in product price. While in the homogeneous model the same $h(P)$ is assumed over all stages, here dependence of $h(P)_t$ on calendar date is allowed for. Seasonal variation in cost or in quality could also be introduced if important.

The Markov chains of the transition probability matrix are no longer completely ergodic and, hence, the steady state probabilities depend upon the initial state. In general, the producer en-

tering the current week with a given flock will face a dilemma he did not face before: (1) to sell the flock before the new season and start a new flock (if it is fairly early in the season, the original framework of the homogeneous model may apply) or (2) to keep the flock until the new season. In the latter case the producer carries a fairly old flock into the new season since, if prices are higher in the new season, it could result in a one-time profit.

An empirical solution for an inter-seasonal model is derived in the empirical results section.

Introduction of intraseasonal price variation. — The homogeneous model assumes the distribution of prices to be independent of past prices. It may be reasonable to assume that price at the n th week depends on price at week $(n - 1)$. In this event, prices within the season might be described by a simple lag model, where current price is a function of price last week. Such dependence of prices over weeks could be described in the continuous case by a stochastic process, such as the Markov process, and in the discrete case by a Markov chain.

In the discrete case the main reason for introducing probabilistic prices is to allow the decision-maker to make use of this additional information about price behavior. To incorporate this price dependence, (60) might be modified as follows:

$$V(X) = W(X) \int_0^\infty P(X) h(P_X/P_{X-1}) dP_X$$

Stochastic Decision Criteria: The Case of Continuous Growth

The model considered in the preceding section is a discrete dynamic decision model. To assume that the decision process followed by an individual producer is discrete would seem reasonable.

$$- C(X), \text{ since } H_X = 0. \tag{71}$$

$$V(x/P_x) = \max \left\{ P_x W(x) - C(x), \int_{P_{x+1}}^\infty V(x + 1/P_{x+1}) h(P_{x+1}/P_x) dP_{x+1} - \pi(P_x) \right\}$$

for $x = X - 1, X - 2, \dots, 7$.

The following differences from the original homogeneous model are noted: (1) In the homogeneous model, the relevant parameters of $h(P)$ were the mean and the variance. In this formulation $h(P)$ is replaced by $h(P_i/P_{i-1})$; an additional parameter is introduced which is the rate of change in prices during the expected life of the flock; and (2) $\pi(P_n)$ replaces π in the homogeneous model

$$\pi(P_n) = \int_{P_7}^\infty V(7/P_7) h(P_7/P_n) dP_7,$$

and

$$\pi = \int_0^\infty \pi(P_n) f(P_n) dP_n$$

where $f(P_n)$ are the steady state probabilities of price transitions.

The solution has not been worked out in detail, but the problem would seem to be solvable by either of the two methods suggested for the homogeneous model. Of course, the dimensions of the problem are increased considerably.

However, to assume that decisions are made only at given arbitrary ages is unrealistic, especially when considering the operation of a system containing a large number of producers.

The analytic solution obtained in the previous section used a continuous price density but did not introduce continuous growth functions explicitly into the decision process. This will be done in the present section.

Again, the assumption is that the system operates for a relatively long time so that it reaches a steady state equilibrium. The decision-making process is Markovian, as described in the previous section, so the steady state probabilities can be obtained from the recursive set of equations (see (58)):

$$q_x = H_{x-1} q_{x-1}, \text{ subject to } \sum_{x=7}^X q_x = 1. \tag{72}$$

Assume a given number of producers (say, a thousand). Let the system operate until equilibrium is reached and then take a cross-section, the fraction of flocks at age x (identical to the fraction of producers with flocks at age x) obtained at this point of time will be equal to q_x (the steady state probability). Hence, the optimal path over time can be projected onto a cross-section taken at a given moment. (This is a direct result of the steady state conditions.) This alternative formulation transforms the problem into one of a continuous growing inventory.

The derivation of the continuous case may be presented as follows: Let

$$K_x = 1 - H_x, \text{ the probability of selling at age } x$$

$$q_{x+\Delta x} = (1 - K_x \Delta x) q_x$$

and

$$\lim_{\Delta x \rightarrow 0} \frac{q_{x+\Delta x} - q_x}{\Delta x} = \frac{dq_x}{dx} = -K_x q_x = -(1 - H_x) q_x.$$

Hence,

$$\frac{dq_x}{dx} = -(1 - H_x) q_x \tag{73}$$

where $-dq_x/dx$ measures the proportion of flocks sold at each age.

Define the profit function over a period from $t_0 = x_0$, (say, age of seven weeks) to $T = X$ (say, age of 12 weeks):

$$\pi = \int_{t_0}^T [R_t - C_t] (1 - H_t) q_t dt \tag{74}$$

where, as in expression (60),

$$C_t = \text{cost of keeping the flock up to age } t = x$$

$$R_t = W_t \int_{P_t^*}^{\infty} P \frac{h(P)}{1 - H_t} dP$$

$$W_t = \text{weight at age } t = x$$

$$\int_{P_t^*}^{\infty} P \frac{h(P)}{1 - H_t} dP = \text{conditional expected price given the condition sell}$$

and

$$P = \text{broiler price.}$$

In the case of the uniform distribution, the computations are quite simple: Define

$$(1 - H_t) = \frac{\bar{P} - P_t^*}{\bar{P} - \underline{P}}$$

and

$$\int_{P_t^*}^{\underline{P}} P h(P) dP = \frac{\bar{P}^2 - (P_t^*)^2}{2(\bar{P} - \underline{P})};$$

then,

$$\int_{P_t^*}^{\underline{P}} P \frac{h(P)}{1 - H_t} = \frac{\bar{P} + P_t^*}{2}$$

and equation (75) may now be rewritten:¹⁷

$$\pi = \int_{t_0}^T \left[C_t \dot{q}_t - W_t \left(\frac{\dot{q}_t^2 \bar{P} - P}{2} + \dot{q}_t \bar{P} \right) \right] dt. \tag{75}$$

¹⁷ A dot denotes differentiation with respect to t .

The maximization problem may be formulated as a simple optimal control problem with q_t as the state variable and \dot{q}_t as the control variable. The solution

is obtained by applying the Pontryagin Maximization Principle (Arrow and Kurz, 1970).

Let the objective function be:

$$\max \int_{t_0}^T \left[C_t \dot{q}_t - W_t \left(\frac{\dot{q}_t^2 \bar{P} - \underline{P}}{2} + \dot{q}_t \bar{P} \right) \right] dt = \max \int_{t_0}^T g(q_t, \dot{q}_t, t) dt^{18} \quad (76)$$

subject to the constraint

$$\dot{q}_t = u_t. \quad (77)$$

Then, the Hamilton-Jacobi equation is

$$\mathcal{H} = g(q_t, u_t, t) + \lambda_t u_t \quad (78)$$

where λ_t is the auxiliary variable associated with the constraint (77) and \mathcal{H} is the current value *Hamiltonian*.

The maximum principle instructs us to maximize equation (78) with respect to u_t . We obtain, therefore,

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial u_t} &= C_t - (\bar{P} - \underline{P}) W_t \frac{u_t}{q_t} \\ &\quad - \bar{P} W_t + \lambda_t = 0. \end{aligned} \quad (79)$$

Hence,

$$\lambda_t = (\bar{P} - \underline{P}) W_t Z_t + \bar{P} W_t - C_t \quad (80)$$

where $Z_t = (d/dt) (\log q_t) = \dot{q}_t/q_t$. Note that λ_t is the marginal contribution of the state variable q_t to the return function $g(q_t, u_t, t)$, and, therefore, is a "price" in an economic sense (see Arrow and Kurz, 1970).

In our case

$$Z_t = \frac{\dot{q}_t}{q_t} = \mathcal{H}_t - 1 = \frac{P_t^* - \bar{P}}{\bar{P} - \underline{P}}$$

and equation (80) becomes

$$\lambda_t = P_t^* W_t - C_t. \quad (81)$$

The marginal contribution of q_t to the current return function is current profit measured at the cutoff price P_t^* .

It can be shown that the rate of change in λ_t may be obtained from:

$$\dot{\lambda}_t = \frac{d\lambda_t}{dt} = - \frac{\partial \mathcal{H}}{\partial q_t}.$$

(See Arrow and Kurz, 1970.) In our case

$$\dot{\lambda}_t = - \frac{1}{2} (\bar{P} - \underline{P}) W_t Z_t^2. \quad (82)$$

Now, differentiating (80) with respect to t and equating to (82), we obtain the following differential equation for Z :

$$\begin{aligned} \frac{1}{2} Z_t^2 + \frac{\dot{W}_t}{W_t} Z_t + \dot{Z}_t \\ + \frac{1}{\bar{P} - \underline{P}} \frac{(\bar{P} \dot{W}_t - \dot{C}_t)}{W_t} = 0. \end{aligned} \quad (83)$$

No simple analytic solution exists for this quadratic differential equation. However, for a given side condition, a solution can be obtained through numerical analysis. Consider the following two border cases:

¹⁸ The problem can be solved equivalently by calculus of variations, which is a special case of the Pontryagin maximization method. This more general presentation is adopted because it lends itself more conveniently to direct economic interpretation.

1. Let the cutoff price P_t^* be equal to \bar{P} everywhere. Note that $P_t^* = \bar{P} \Rightarrow Z_t = 0$, that is, keep the flock at all prices. Equation (83) reduces to the familiar form: $\bar{P} \dot{W}_t = \dot{C}_t$. The flock is kept up to the age where value of marginal growth in output is equal to value of marginal growth in input. This boundary condition can be interpreted in the following way: It is equal to zero until sending the flock to market, then suddenly drops to $-\infty$. This, in fact, is the case in the determi-

nistic world where $\bar{P} = \underline{P} = P$.

2. Let the cutoff price P_t^* equal \underline{P} everywhere. Hence,

$$P_t^* = \underline{P} \Rightarrow Z_t = \frac{\dot{q}_t}{q_t} = -1$$

which implies

$$q_t = e^{-t}.$$

Z_t is constant with an absolute unit value.

SOME EMPIRICAL RESULTS

The main objective of this section is to present and examine some empirical results based on formulations in the last section. Data in all cases refer to Israeli broiler producing firms.

Broiler production in Israel is carried out by nonintegrated farms, where the marketing decisions are also made.¹⁹ The typical poultry producer raises both layers and broilers, but specific equip-

ment is used for the separate broiler and layer enterprises. The smallest unit of production is the flock, and, because interest here is in a flock operation at a given capacity, production and decision variables are defined on a per-bird basis. Certain relevant empirical functions are derived first. Then, some numerical results are generated for the sequential stochastic decision model.

The Empirical Functions

The weight-feed relations

Two sets of data collected in Israel are available for analysis:²⁰

Data Set I: This set includes data collected in a 1961 survey of 22 flocks of broilers. Weekly observations are available from the day the chickens hatched to the day they were sent

to the district slaughterhouses. The survey was not designed to give a representative sample of broiler producers in Israel (most of the data were collected in the regional district of Shaar Hane-gov). We shall refer to this set of data as "field data."

¹⁹ The concept of an individual producer may apply both to a farm in a cooperative village "moshav" or a broiler production industry in a collective village "kibbutz."

²⁰ The data were collected for different purposes as part of a comprehensive project. See Mundlak (1964).

Data Set II: The results of two broiler random tests conducted in Israel (1959-60) at the Experiment Station at Aco. We shall refer to this set of data as "Aco data."

It seems advantageous to use first the Aco data. In the randomized experiments generating the data, all flocks were under the same management, receiving the same treatment, and thereby reducing differences between flocks.²¹ Therefore, it would appear reasonable to estimate average relations as described in equations (4), (5.1) and (17.1).

The parameters of equation (4), α_0 and α_1 , were estimated from the log-reciprocal transformation:²²

$$\log_e W = 8.5467 - 12.356 \frac{1}{x} \quad (84)$$

(29.34) $R^2 = .869.$

Direct estimation of the parameters γ and β_1 in equation (5.1) yields

$$\log_e F = 5.07 + 1.477 \log_e x \quad (85)$$

$$- 2.959 \frac{1}{x} \quad (38) \quad R^2 = .921.$$

The coefficients in (85) are not plausible, and t values are low. This may well be due to multicollinearity as the correlation between x and $1/x$ is $-.99851$. However, indirect estimates of the parameters are possible from equation (17.1):

$$\log_e W = .568 + .911 \log_e F \quad (29.9) \quad (86)$$

$$- .306 \log_e x \quad (5.39) \quad R^2 = .984.$$

From equation (17.1), $\delta_1 = \alpha_1/\beta_1$ and $\delta_2 = -\delta_1 \hat{\gamma}$. Accordingly,

$$\hat{\gamma} = -\frac{\hat{\delta}_2}{\hat{\delta}_1} = .3365 \text{ and } \hat{\beta}_1 = 13.569.$$

If the hypothesis $\alpha_1 = \beta_1 = g$ is acceptable, one may estimate γ from:

$$\log_e \left(\frac{F}{W} \right) = .206 + .466 \log_e x. \quad (87)$$

(28.12) $R^2 = .859.$

The estimate $\hat{\gamma} = .466$ is adopted and imposed on the feed equation giving:

$$(\log_e F - .466 \log_e x) \quad (88)$$

$$= 8.341 - 12.364 \frac{1}{x} \quad (28.94) \quad R^2 = .866.$$

Hence, under the assumption of equality between the growth rates, $\alpha_1 = \beta_1$, $\hat{g} = 12.4$.

Because our sample included 33 flocks, different growth rates can be allowed in the method suggested in equations (8) and (9), where the Y 's and the Z 's are dummy variables allowing for a flock effect on the growth rate. The results allowing for flock effect are given in table 2.

The agreement between the estimated coefficients in equations (84) and (88) provides some support for the claim that our assumptions hold. One may question whether this is an appropriate test because all estimates are based on the

²¹ These differences amount mainly to source of supply or breed.

²² Figures in parentheses under the coefficients are ratios of coefficients to their standard errors.

TABLE 2
ESTIMATES OF $\hat{\alpha}_i$ AND $\hat{\beta}_i$
ALLOWING FOR FLOCK EFFECT

Flock	$\hat{\alpha}_i$	$\hat{\beta}_i$	$\hat{\alpha}_i - \hat{\beta}_i$
1.....	13.247	13.430	-.183
2.....	13.253	13.262	-.009
3.....	12.201	12.388*	-.187
4.....	12.417*	12.566	-.149
5.....	12.844	12.918	-.075
6.....	11.669	11.704	-.035
7.....	12.685	12.673	.011
8.....	12.016	12.301*	-.285
9.....	11.291	11.434	-.142
10.....	12.311*	12.665	-.354
11.....	12.708	12.766	-.058
12.....	12.346*	12.442	-.096
13.....	12.404*	12.529	-.125
14.....	12.457*	12.728	-.271
15.....	12.412*	12.720	-.308
16.....	12.627	12.349*	.277
17.....	12.111	11.838	.273
18.....	12.577*	12.508	.068
19.....	11.646	11.319	.327
20.....	12.134	12.056	.078
21.....	12.437*	12.267*	.170
22.....	12.897	12.647	.251
23.....	12.501*	12.294*	.207
24.....	12.086	12.021	.065
25.....	12.438	13.135	.303
26.....	12.699	12.692	.007
27.....	12.049	11.987	.063
28.....	12.528*	12.637	-.109
29.....	12.592	12.441	.041
30.....	12.370*	12.492	-.122
31.....	11.277	11.173	.104
32.....	11.115	11.131	-.016
33.....	12.409	12.323	.086
R ²981	.985	

* The ratio of the flock coefficient to its standard deviation is less than one.

Source: The coefficients were estimated from the relations defined in footnote 2, page 6.

$$\text{Log } W_i = \alpha_0 - a_0 \frac{1}{x} - a_1 Y_{1i} \frac{1}{x}, \bar{F}_i = \beta_0 - b_0 \frac{1}{x} - b_1 Z_{1i} \frac{1}{x}$$

where
 Y_{1i} = dummy variable allowing for flock effect
 Z_{1i} = dummy variable allowing for flock effect
 $\bar{F}_i = (\text{Log } F_i - 4666 \text{ Log }_e x)$
 $\alpha_{1i} = a_0 + a_1 Y_{1i}$

and
 $\beta_{1i} = b_0 + b_1 Z_{1i}$.
 The coefficients were computed relative to the 33rd flock.

same set of data, and the circularity argument cannot be disregarded. Equality was imposed and then found to be true. Nevertheless, the two methods of estimation of g —one through estimating $\hat{\alpha}_i$ and the other through estimating $\hat{\beta}_i$ —are not computationally identical.

A more rigorous test is to test our

hypothesis on a different set of data. Accordingly, the same procedure was applied to the set of field data, giving practically the same estimates of γ and g . After imposing the hypothesis, the estimate of γ is $\hat{\gamma} = .461$ and the estimate of g is $\hat{g} = 83.2$. Note that for comparison with Aco data \hat{g} must be divided by 7 since x is measured in weeks in the Aco data and in days in the field data. Adjusting accordingly, $\hat{g}/7 = 11.9$ compared to $\hat{g} = 12.4$ from Aco data.

In the numerical results summarized subsequently in the next section, a basic model is assumed as a point of departure in which the parameters of the weight-feed relation take the values.

$$\begin{aligned} \hat{g} &= \hat{\alpha}_1 = \hat{\beta}_1 = 86.5,^{23} \\ \hat{\gamma} &= .466, \\ e^{\hat{\alpha}_0} &= 8.547, \\ e^{\hat{\beta}_0} &= 7.410 \end{aligned}$$

and the empirical cost function is:

$$C(x) = C_0 + P_f F(x) = .859 + .26 F(x) \quad (89)$$

The quality function

As previously indicated, quality is regarded here as related to age of flock. Equation (20), assumed to represent this relation, is repeated here for convenience.

$$Q = \frac{x}{\Psi} e^{1-x/\Psi} \quad (20)$$

where x is age of flock in days. The parameter Ψ determines the age at which Q reaches its maximum. In the "basic model," maximum quality is assumed to be reached at 63 days. Accordingly, for the empirical analysis, $\hat{\Psi} = 63$.

²³ Where age of flock is measured in days.

The price density function $h(P)$

Monthly data on broiler prices²⁴ in Israel for the years 1960-61 were used in deriving empirical density functions. These prices were as follows:

1960		1961	
January	1.281	January	1.386
February	1.245	February	1.250
March	1.461	March	1.320
April	1.539	April	1.599
May	1.679	May	1.901
June	1.754	June	2.172
July	1.740	July	2.030
August	1.819	August	1.691
September	1.887	September	2.002
October	1.779	October	1.401
November	1.519	November	1.473
December	1.456	December	1.529

Two seasons are distinguished:²⁵

- Season 1: October-March
- Season 2: April-September.

Empirical distributions were derived corresponding to two underlying distributions of prices: the normal distribution and the uniform distribution.

For the normal distribution, maximum likelihood estimates of means and the variances were computed:²⁶

$$\begin{aligned}\hat{\mu}_1 &= 1.425 & \hat{\sigma}_1^2 &= .0861 \\ \hat{\mu}_2 &= 1.818 & \hat{\sigma}_2^2 &= .1255.\end{aligned}$$

The arrays of prices P_k^j were then derived from:

$$Z_k = \frac{P_k^j - \hat{\mu}_j}{\hat{\sigma}_j} \Rightarrow P_k^j = Z_k \hat{\sigma}_j + \hat{\mu}_j.$$

Twenty intervals ($k = 1, 2, \dots, 20$) were selected such that the rectangles of the histogram each have the same area, equal to .05. The derived price array for each of the two seasons is given in Appendix table A-1.

Use of the normal distribution can perhaps be more easily defended on the basis of the central limit theorem. On the other hand, the uniform distribution is convenient for computational purposes. And though it may not describe reality as well as does the normal distribution, it is useful in demonstrating the working of the system.

For the uniform distribution,

$$h(P) \begin{cases} = \frac{1}{\bar{P} - P} & P \leq P \leq \bar{P} \\ = 0 & \text{elsewhere} \end{cases}$$

²⁴ Prices are in Israeli pounds (IL). Approximate conversion rate: three IL = \$1.00.

²⁵ This break into seasons was adopted so that interseasonal price variation was greater than intraseasonal price variation.

²⁶ Because the original data, which are monthly averages, do not suit our purpose, weekly distribution parameters were derived, assuming that quantities of broilers sold during the month were spread equally over weeks. Therefore, the average price per month may be regarded as an unbiased estimate of expected price per week. To derive an estimate of the variance of prices on a weekly basis, it can be shown that, for equally distributed weekly quantities sold, the variance estimates may be obtained from:

$$\sigma_i^2 = 4 \sigma_p^2,$$

where

$$\begin{aligned}j &= \text{season}; j = 1, 2 \\ \sigma_{p_j}^2 &= \text{variance of the original monthly averages}\end{aligned}$$

and

$$\sigma_i^2 = \text{variance on weekly basis.}$$

$$H(P) \begin{cases} = 0 & P < \underline{P} \\ = \frac{P - \underline{P}}{\bar{P} - \underline{P}} & P \leq P \leq \bar{P} \\ = 1 & \bar{P} < P \end{cases}$$

	\underline{P}	\bar{P}
Season 1:	.978	2.046
Season 2:	1.223	2.488

The arrays of prices P_k^j are then derived from:

$$P_k^j = (\bar{P} - \underline{P}) H(P_k) + \underline{P}$$

where $H(P_k) = .025, .075, .125, \dots, .975$.

The derived price arrays for the two seasons are given in Appendix table A-1.

where \bar{P} is the upper extreme price and \underline{P} is the lower extreme price. Similar adjustments were made to transform the monthly data to a weekly basis.²⁷

The corresponding maximum likelihood estimates are:

Discrete Sequential Decision Models

In this discussion of empirical results attention focuses on optimal policies for the system (and convergence of solution to optimal), sensitivity of the optimal solution to changes in parameters of the model, comparison between the analytic and the functional equation solution, and extension of the homogeneous model to an interseasonal model. In generating

the numerical results, use is made of the empirical functions presented above.

The functional equation

It is first necessary to write the basic functional equation (46) in a form manageable for computational purposes. Accordingly, continuous prices assumed

²⁷ Again, assuming the same average, we have the following two equations from which to solve for \bar{P} and \underline{P} :

$$(i) \quad \frac{\bar{Y} + \underline{Y}}{2} = \mu = \frac{\bar{P} + \underline{P}}{2}$$

where

- \bar{Y} = monthly price averages
- \bar{Y} = monthly upper price extreme
- \underline{Y} = monthly lower price extreme
- \bar{P} = weekly upper price extreme.

and

\underline{P} = weekly lower price extreme.

$$\frac{(\bar{Y} - \underline{Y})^2}{12} = \sigma_{p_i}^2, \text{ and again assuming equally}$$

(ii) distributed weekly quantities sold,

$$\sigma_{p_i}^2 = \frac{\sigma_i^2}{4} = \sigma_i^2 \Rightarrow \frac{(\bar{Y} - \underline{Y})^2}{12} \cdot 4 = \frac{(\bar{P} - \underline{P})^2}{12}$$

From (ii) we have

$$(iii) \quad (\bar{Y} - \underline{Y}) 2 = \bar{P} - \underline{P}$$

Equations (i) and (iii) are solved for the two unknowns, \bar{P} and \underline{P} .

TABLE 3
OPTIMAL POLICIES

Season	Normal distribution					Number of iterations	Uniform distribution					
	Age (weeks)						Age (weeks)					
	7	8	9	10	11		7	8	9	10	11	
	IOP (H_x)						IOP (H_x)					
1.....	.85	.70	.60	.50	.35	9	.75	.65	.55	.50	.40	8
2.....	.55	.50	.45	.40	.30	7	.55	.50	.45	.40	.35	7

in (46) are approximated by a discrete array. A convenient way is to break $h(P)$ into a histogram of K equal probability rectangles. This results in an array of K

prices and an associated set of discrete probabilities.

This approximation allows us to rewrite equation (46) as:

$$g_n(x) = \max \left[\begin{array}{l} S: W(x) P_k - C(x) + E_P(g_{n-1}(7)) \\ K: E_P(g_{n-1}(x+1)) \end{array} \right]$$

for $k = 1, 2, \dots, 20$ and $x = 7, 8, \dots, 11$.

$$E_P(g_{n-1}(x)) = \frac{1}{20} \sum_{k=1}^{20} g_{n-1}(x) \quad (90)$$

for $x = 7, 8, \dots, 11$.

$$E_P(g_n(12)) = W(12) \left[\frac{1}{20} \sum_{k=1}^{20} P_k \right] - C(12) + E_P(g_{n-1}(7)).$$

Optimal policies

Solutions are presented for the basic model under four different assumptions about the distribution of prices:²⁸

Model 1.1: for season 1 assuming a normal distribution of prices.

Model 1.2: for season 1 assuming a uniform distribution of prices.

Model 2.1: for season 2 assuming a normal distribution of prices.

Model 2.2: for season 2 assuming a uniform distribution of prices.

Results are presented in table 3 and figure 1. For a given season the difference between assuming a normal or a uniform distribution of prices is not considerable. The dispersion of cutoff prices is less with the uniform than with the normal distribution. Nevertheless, the differences are minor. This may suggest use of the uniform distribution, especially where computational considerations are important.

²⁸ For the numerical values of the basic growth and quality parameters, see page 33.

I. Season 1

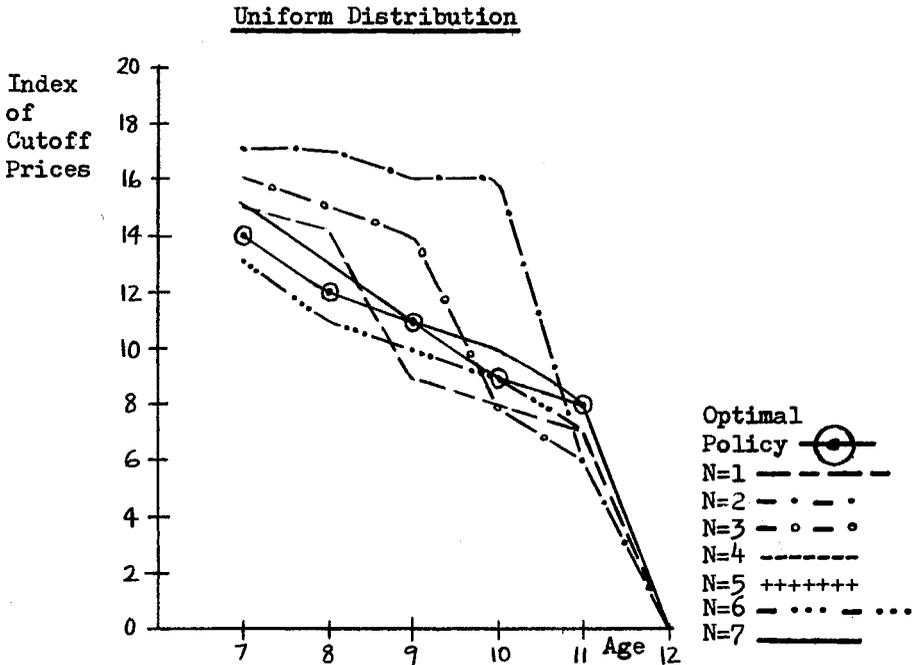
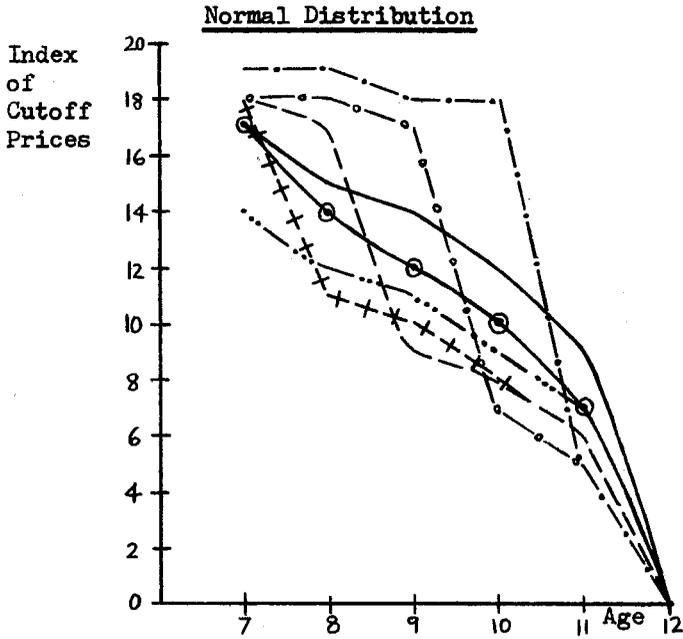


FIGURE 1—(CONTINUED ON NEXT PAGE)

II. Season 2

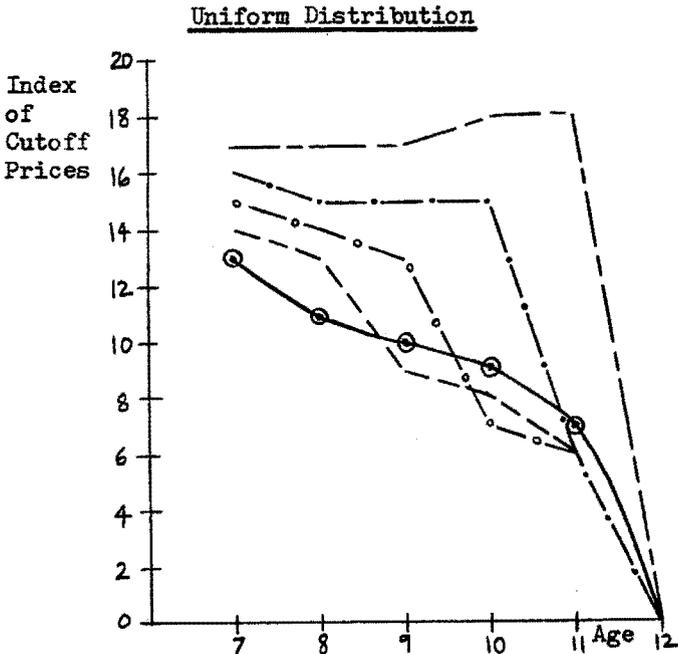
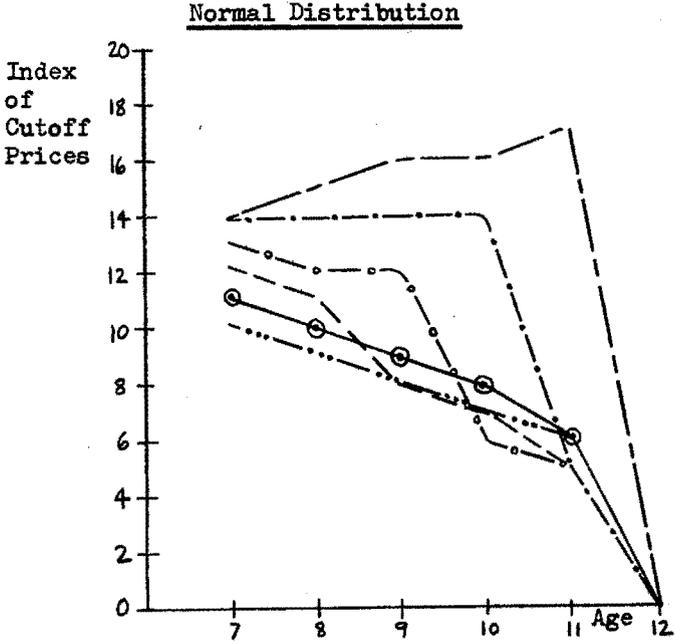


Fig. 1. Convergence to optimal policy—homogeneous models. (Prices corresponding to the index of cutoff prices for each season for both the normal and uniform distribution are tabulated in Appendix Table A-1. *N* denotes order of iteration.)

TABLE 4
EFFECTS OF CHANGES IN THE VARIANCE OF PRICES

		Age (weeks)					
		7	8	9	10	11	12
I		Number of iterations 9, $\pi = .1667$ IL					
$V(P) = .0215$	H_x	.9	.65	.55	.45	.35	0
$\bar{P} = 1.779$	$1 - H_x$.1	.35	.34	.55	.65	1
$\underline{P} = 1.245$	q	.3331	.2998	.1949	.1072	.0482	.0168
$E(P) = 1.512$	$P^*(x)$	1.7256	1.5921	1.5387	1.4853	1.4319	1.245
	$E(x)$.0191	.1339	.2496	.3661	.4645	.5879
II		Number of iterations 8, $\pi = .2293$ IL					
$V(P) = .0861$	H_x	.7	.6	.55	.45	.4	0
$\bar{P} = 2.046$	$1 - H_x$.3	.4	.45	.5	.6	1
$\underline{P} = .978$	q	.3980	.2786	.1671	.0919	.0460	.0184
$E(P) = 1.512$	$P^*(x)$	1.726	1.619	1.565	1.512	1.405	.978
	$E(x)$.0908	.2168	.3366	.4421	.5473	.5879
III		Number of iterations 6, $\pi = .2965$ IL					
$V(P) = .1935$	H_x	.65	.6	.55	.5	.4	0
$\bar{P} = 2.313$	$1 - H_x$.35	.4	.45	.5	.6	1
$\underline{P} = .711$	q	.4159	.2703	.1622	.0892	.0446	.0178
$E(P) = 1.512$	$P^*(x)$	1.7523	1.6722	1.5921	1.512	1.3518	.711
	$E(x)$.1508	.2871	.4221	.5414	.6499	.5879

Comparing the two seasons, optimal policy appears to call for selling at earlier ages in season 2 than in season 1, that is, cutoff prices in season 2 are lower than in season 1. In the distribution of prices for the two seasons, both the expected value and the variance of prices are higher in season 2 than in season 1. Other things equal, one would expect to sell at earlier ages in the season for which expected value of price is higher. As for the effect of the variance, it is of interest to observe the effects of changes in variance which are reported and examined below.

Figure 1 also depicts convergence of the solution toward the invariant optimal policy (IOP). Convergence is achieved in a relatively small number of iterations (7 to 9). The number of iterations decreases as we move from season 1 to season 2. Finally, the IOP cutoff prices are described by a negatively

sloped sigmoid curve. This characteristic may be a result of the structural relations.

Effects of changes in the variance of prices

Two results are associated with an increase in the variance of prices $V(P)$: First, sales tend to get dispersed over age groups. The lower extreme, that is, $V(P) = 0$, corresponds to the deterministic case, where all sales are concentrated at a given age. Second, π increases as $V(P)$ increases. The second result is surprising because one expects an increase in variance to appear as an increase in costs, but the result is the opposite.

To illustrate, three values are assumed for the variance of the uniform distribution for season 1, the expected price remaining at 1.512. Results are reported in table 4 and figure 2.

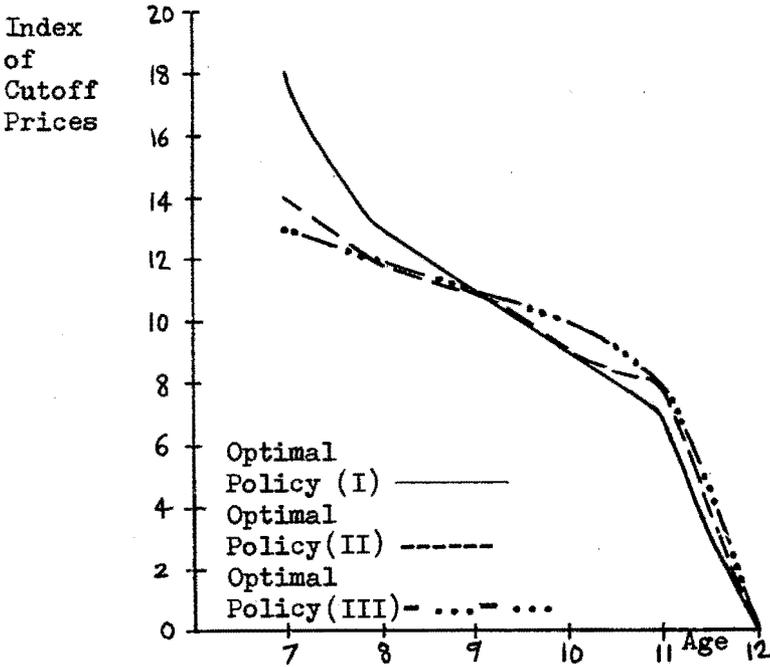


Fig. 2. Effects of changes in the variance of prices: I.O.P. cutoff prices compared. (Prices corresponding to the index of cutoff prices are tabulated in Appendix Table A-1.)

From table 4 it appears that the probability of selling at younger ages increases as $V(P)$ increases. This is shown by $(1 - H_x)$, the probability to sell given the 20 possible prices at each age. Furthermore, we note that $\pi_1 < \pi_2 < \pi_3$. This can be explained by the fact that $E(x)$ increases as the variance increases. The conditional expected net returns from sales increase since they are affected by the more extreme higher prices above the cutoff price and not affected by the extremes below the cutoff price. In economic terms the producer is able to take advantage of the information about $V(P)$ in a world in which risk increases (where increase in risk is implied by the increase in $V(P)$).

Policy implications

The sequential stochastic model can supply the broiler breeder and the pub-

lic policy-maker with important information also. Following the same lines as in the deterministic case, the implications of improving the growth, maintenance feed, and quality parameters and of changes in the price of feed are evaluated. But while in the deterministic case the effect on the unique optimal marketing age was considered, the effect on the IOP is of interest in the present case.

The effects of variation in the parameters and the price of feed on the IOP (the vector H_x) are reported in table 5. Parameters are varied one at a time, the remaining parameters held at their respective values in the basic model. Output prices refer to season 1 and are assumed to be normally distributed.

The growth coefficient.—The following range of values of the growth coefficient ($\alpha_1 = \beta_1 = g$) is considered: $g = 75, 80, 85, 90, 95, \text{ and } 100$.

TABLE 5

SENSITIVITY OF THE INVARIANT OPTIMAL POLICIES (H_x) TO VARIATION IN g , γ , Ψ , AND P_f (NORMAL DISTRIBUTION OF BROILER PRICES)

Change in parameter	Age (weeks)					Number of iterations
	7	8	9	10	11	
Basic model*	.85	.70	.60	.50	.35	9
	H_x					
g						
75	.60	.55	.45	.40	.30	8
80	.70	.60	.55	.45	.35	9
85	.80	.70	.60	.50	.35	9
90	.90	.75	.65	.55	.40	11
95	.95	.85	.70	.60	.40	12
100	.95	.85	.75	.60	.45	15
γ						
.35	.75	.65	.55	.45	.35	8
.40	.80	.65	.55	.50	.35	10
.45	.80	.70	.60	.50	.35	9
.50	.85	.75	.60	.50	.35	9
.55	.90	.75	.65	.55	.40	11
Ψ						
42	.70	.55	.45	.35	.25	8
49	.75	.68	.50	.40	.30	8
56	.80	.65	.55	.45	.35	9
63	.85	.70	.60	.50	.35	9
70	.90	.85	.75	.65	.50	10
P_f						
.20	.75	.65	.55	.45	.35	10
.25	.85	.70	.60	.50	.35	9
.30	.85	.75	.65	.50	.40	11
.35	.90	.75	.65	.55	.40	10

* The basic model parameters are: $\alpha_0 = 8.5467$, $\beta_0 = 7.410$, $\alpha_1 = \beta_1 = g = 86.5$, $\gamma = .466$, $\Psi = 63$, and $P_f = .260$.

The growth coefficient determines the inflection point. Hence, the lower the g , the faster the growth. This explains the tendency to market the flock at lower ages for lower values of this parameter. Note that cutoff prices are more sensitive to changes in the growth coefficient at younger ages than at older ages. This results from two forces operating in the same direction: (1) the recursiveness of the system which results in an accumulative impact of changes recursively from older to younger ages and (2) for the age range considered, the effect of diminishing marginal rate of growth increases as g increases.

The maintenance feed coefficient.

—The maintenance feed coefficient (γ) is assigned values: $\gamma = .35, .40, .45, .50$, and $.55$. Unlike the growth coefficients, the maintenance coefficient enters only the feed equation. An increase in γ raises both the marginal and the average feed consumption. In the deterministic decision model considering one flock, the optimal marketing age decreases as γ increases since marginal cost is increased. Here we have the opposite result. In general, flocks would be marketed at earlier ages for *lower* γ . Contributing to this is the fact that optimality conditions in the sequential

TABLE 6
 SENSITIVITY OF INVARIANT OPTIMAL POLICIES (H_z) TO VARIATION
 IN g , γ , Ψ , AND P_f (UNIFORM DISTRIBUTION OF BROILER PRICES)

Change in parameter	Age (weeks)					Number of iterations
	7	8	9	10	11	
Basic model*	.75	.65	.55	.50	.40	9
	H_z					
g						
75	.50	.50	.45	.40	.35	9
80	.60	.55	.50	.45	.35	7
85	.70	.60	.55	.45	.40	10
90	.80	.70	.60	.50	.40	8
95	.90	.75	.65	.55	.40	11
100	1.00	.80	.65	.55	.45	10
γ						
.35	.65	.55	.50	.45	.35	8
.40	.70	.60	.55	.45	.35	8
.45	.70	.60	.55	.50	.40	10
.50	.75	.65	.55	.50	.40	9
.55	.80	.70	.60	.50	.40	9
Ψ						
42	.60	.50	.45	.35	.30	8
49	.65	.55	.45	.40	.35	9
56	.70	.60	.50	.45	.35	8
63	.75	.65	.55	.50	.40	9
70	.80	.70	.60	.50	.40	9
P_f						
.20	.65	.60	.55	.45	.35	8
.25	.70	.65	.55	.50	.40	12
.30	.75	.65	.60	.50	.40	11
.35	.85	.70	.60	.50	.40	9

* The basic model parameters are: $\alpha_0 = 8.5467$, $\beta_0 = 7.410$, $\alpha_1 = \beta_1 = g = 86.5$, $\gamma = .466$, $\Psi = 63$, and $P_f = .260$.

stochastic model involve values of expected net returns ($E(x)$ and π) which presumably depend negatively on γ . This is apparent from equation (60) where it can be verified that a smaller γ results generally in higher cutoff prices. The decision does not involve just equating marginal cost to the value of marginal product. The comparison is between different policies. And because in this context the whole process is relevant, we are concerned with expected returns.

Again, the sensitivity of cutoff prices to changes in the coefficient is higher for younger ages than for older ages. The factor of recursiveness operates here in

the same way as for the growth coefficient. A second relevant factor is the positive relation between γ and marginal feed consumption.

The quality coefficient.—The following alternative values for the quality coefficient (Ψ) are considered: $\Psi = 42$, 49, 56, 63, and 70.

As expected, flocks would be marketed at earlier ages for smaller Ψ 's. The producer takes advantage of the fact that the bird reaches top quality at an earlier age. It appears, however, that the sensitivity of cutoff prices to variation in the quality coefficient is fairly uniform over ages. This is a result of two offset-

ting factors. The recursiveness of the system works in the same direction as in the previous cases. But the effect of increases in the quality parameter is increasing weight at older ages, and this has the opposite effect on cutoff prices.

Price of feed.—Price of feed has been varied as follows: $P_f = .20, .25, .30,$ and $.35$.

The direction of influence of increasing feed price is the same as for increasing γ . Price of feed is controlled by the government in Israel and used as a tool to subsidize or to tax producers. Since the price of feed is used as a policy tool, it is important to be able to evaluate the impact of changes in the price of feed on net returns of the producer.

A similar sensitivity exercise for identical patterns of variation in parameters was conducted also for an assumed uniform distribution of output prices. The results are reported in table 6. Essentially, the results are similar, but there are some differences in the sensitivity at

earlier ages. As opposed to the uniform distribution, the normal distribution assumes extreme values of prices to occur with a very low probability. That the difference in price distribution manifests itself in differences in sensitivity only for lower ages is attributable to the fact that at older ages the effect of low possible selling prices is overshadowed by the dominant influence of termination age.

Computation based on the analytic solution

The numerical results in table 7 are based on direct solution of the set of equations (60), given the definition of π in equation (52). The uniform distribution is assumed since it is convenient for computational purposes. The main purpose is to demonstrate an alternative computational method and to show that the results obtained are the same as those from the functional equation method.

Using the following derived results for the uniform distribution

$$\int_{P^*(x)}^{\infty} P h(P) dP = \frac{\bar{P}^2 - (P^*(x))^2}{2(\bar{P} - \underline{P})}, H_x = \int_0^{P^*(x)} h(P) dP = \frac{P^*(x) - \underline{P}}{\bar{P} - \underline{P}},$$

$$q_7 = \frac{1}{1 + H_7 + H_7 H_8 + \dots + H_7 H_8 \dots H_X}, \text{ and } q_x = H_{x-1} q_{x-1}$$

for $x = 8, 9, \dots, X,$

equations (52) and (60) may be rewritten:

$$\pi = \sum_{x=7}^{12} \left[\frac{\bar{P}^2 - (P^*(x))^2}{2(\bar{P} - \underline{P})} W(x) - \frac{P - P^*(x)}{(\bar{P} - \underline{P})} C(x) \right] q_x, \tag{91}$$

$$P^*(x) = \frac{1}{W_x} \left\{ W(x+1) \frac{\bar{P}^2 - (P^*(x+1))^2}{2(\bar{P} - \underline{P})} - \frac{\bar{P} - P^*(x+1)}{\bar{P} - \underline{P}} C(x+1) \right. \tag{92}$$

$$\left. - \pi + \frac{P^*(x+1) - \underline{P}}{\bar{P} - \underline{P}} [P^*(x+1) W(x+1) - C(x+1)] + C(x) \right\}$$

for $x = 11, 10, 9, 8, 7,$ and where $H_{12} = 0$.

TABLE 7
DIRECT SOLUTION

Iteration	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	π
0.....	.500	.500	.500	.500	.500	0	.2146
1.....	.771	.660	.575	.497	.393	0	.2296
2.....	.736	.637	.561	.494	.384	0	.2300
3.....	.733	.634	.557	.490	.384	0	.2293
IOP: functional equation solution.....	.75	.65	.55	.50	.40	0	.2293

The six equations in (91) and (92) are solved for the six unknowns— π , $P^*(11)$, $P^*(10)$, $P^*(9)$, $P^*(8)$, and $P^*(7)$. As suggested on page 26, one starts from an initial π computed for all P^* 's equal to the corresponding expected prices, then solves recursively for $P^*(11)$, $P^*(10)$, ..., $P^*(7)$. The amount of computation is relatively small and can be performed by desk calculator.²⁹

An interseasonal model

Two seasons are defined—one corresponding to the first 26 weeks and the second corresponding to the last 26 weeks.

Using the same approximation as in (90), equation (63) may be written:

$$g_n(x, t) = \max \tag{93}$$

$$\left[\begin{array}{l} S: R(x, t) - C(x) + E_p[g_{n-1}(7, t - 1)] \\ H: E_p[g_{n-1}(x + 1, t - 1)] \end{array} \right]$$

$$\begin{aligned} &\text{for } k = 1, 2, \dots, 20; \\ &x = 7, 8, \dots, 11; \\ &t = \begin{cases} 1, 2, \dots, 26 & \text{(season 1)} \\ 27, 28, \dots, 52 & \text{(season 2)}. \end{cases} \end{aligned}$$

$E_p[g_{n-1}(x, t)]$ and $E_p[g_{n-1}(X, t)]$ are defined in (90), except that different price distributions are assumed for seasons 1 and 2. This means that seasonality is introduced only through the price of broilers. Results are presented only for the normal distribution of prices.

The solution involves the same recursive method as the homogeneous model, except that the calendar date appears explicitly. As a result, the solution for IOP contains 52 policies, one for each week. Convergence is verified when two consecutive years have identical solutions.

Because of the simplicity of a model assuming only two seasons, convergence was very fast. Solution was obtained in

²⁹ In the case of the normal distribution:

Set $p = \frac{P - \mu}{\sigma_p}$, p is price as a standard normal variable.

Then

$$\int_{P^*(x)}^{\infty} p e^{-(1/2)p^2} dp = \int_{P^*(x)}^{\infty} e^{-u} du = e^{-(1/2)P^*(x)^2}$$

for which tables are available where $u = (1/2)p^2$, and

$$(1 - H_x) = \int_{P^*(x)}^{\infty} e^{-(1/2)p^2} dp$$

for which tables are available.

TABLE 8
INVARIANT OPTIMAL POLICIES FOR THE SEASONAL MODEL
(NORMAL PRICE DISTRIBUTION)

Week (t)	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}
1-19.....	.85	.70	.60	.50	.35	0
20.....	.85	.75	.65	.50	.40	0
21.....	.80	.70	.60	.50	.40	0
22.....	.80	.65	.60	.50	.35	0
23.....	.85	.70	.55	.45	.35	0
24.....	.85	.75	.60	.40	.30	0
25.....	.90	.85	.75	.65	.30	0
26.....	.95	.90	.85	.85	.75	0
27-50.....	.55	.50	.45	.40	.30	0
51.....	.50	.45	.40	.35	.30	0
52.....	.40	.25	.20	.15	.10	0

the second year and verified in the third. Figure 3 and table 8 present the optimal policy for the entire year.

For $t = 1$ to $t = 19$, a homogeneous *intra*seasonal IOP for season 1 is apparent. For $t = 20$ up to $t = 26$, the IOP reflects the transition from season 1 to season 2. As one would expect, the producer keeps his flocks for an additional time in order to make extra profit by

entering season 2 with relatively older flocks. $t = 27$ to $t = 50$ represents a homogeneous *intra*seasonal IOP for season 2. Finally, $t = 51$ and $t = 52$ represent the transition period from season 2 to season 1. This is shorter than the other transition period, reflecting a policy for the producer of disposing of as many flocks as he can before entering the low-price season.

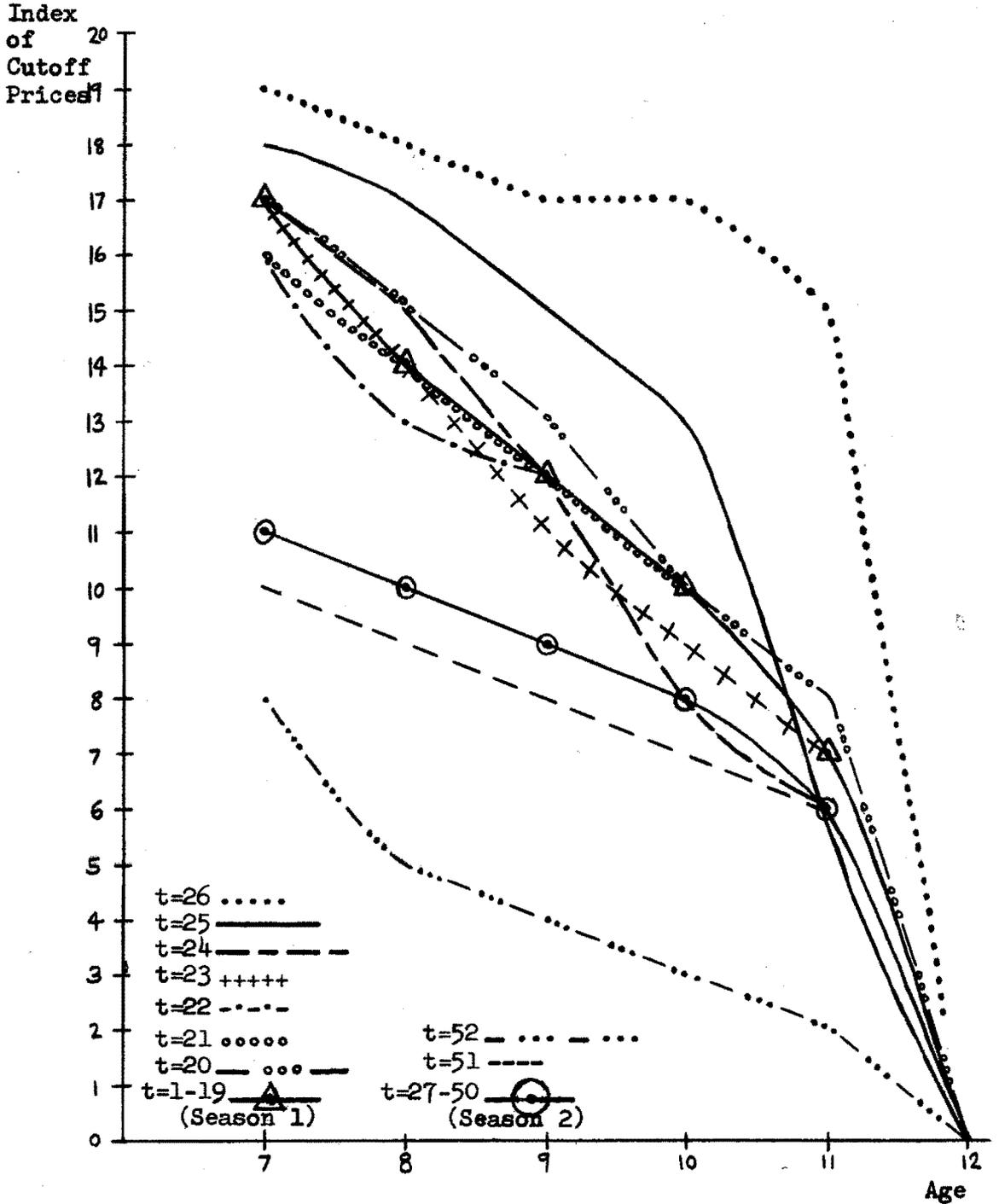


Fig. 3. Invariant optimal policies for the seasonal model. (Prices corresponding to the index of cutoff prices are tabulated in Appendix Table A-1. t denotes week. Season 1: October-March [$t = 1, 2, \dots, 26$]; season 2: April-September [$t = 27, 28, \dots, 52$].)

APPENDIX TABLE A-1
COMPUTED BROILER PRICES, BY SEASON

Cumulative density		Normal distribution		Uniform distribution	
Index	$H(P)$	Season 1	Season 2	Season 1	Season 2
IL (Israeli pounds)					
1	.05	.850	1.124	1.005	1.255
2	.10	1.003	1.308	1.058	1.318
3	.15	1.088	1.411	1.112	1.381
4	.20	1.149	1.485	1.165	1.444
5	.25	1.202	1.549	1.218	1.508
6	.30	1.249	1.605	1.272	1.571
7	.35	1.298	1.659	1.325	1.634
8	.40	1.331	1.705	1.379	1.698
9	.45	1.369	1.751	1.432	1.761
10	.50	1.407	1.797	1.485	1.824
11	.55	1.443	1.839	1.539	1.887
12	.60	1.481	1.885	1.592	1.950
13	.65	1.519	1.931	1.646	2.014
14	.70	1.557	1.977	1.699	2.077
15	.75	1.601	2.031	1.752	2.140
16	.80	1.648	2.087	1.806	2.203
17	.85	1.701	2.151	1.859	2.267
18	.90	1.762	2.225	1.913	2.330
19	.95	1.842	2.328	1.966	2.393
20	1.00	2.000	2.512	2.019	2.456

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